XLVII. *Further Calculations concerning the Momentum of Progressive Waves.* By Lord RAYLEIGH, *O.M., F.R.S.**

THE question of the momentum of waves in fluid is of interest and has given rise to some difference of opinion. In a paper published several years ago † I gave an approximate treatment of some problems of this kind. For a fluid moving in one dimension for which the relation between pressure and density is expressed by

$$p=f(\rho), \quad \dots \quad (1)$$

it appeared that the momentum of a progressive wave of mean density equal to that of the undisturbed fluid is given by

$$\left\{\frac{\rho_0 f''(\rho_0)}{4a^3}+\frac{1}{2a}\right\}\times \text{total energy}, \quad \dots \quad (2)$$

in which ρ_0 is the undisturbed density and a the velocity of propagation. The momentum is reckoned positive when it is in the direction of wave-propagation.

For the "adiabatic" law, viz.:

$$p/p_0=(\rho/\rho_0)^\gamma, \quad \dots \quad (3)$$

$$f'(\rho_0)=\frac{\gamma p_0}{\rho_0}=a^2, \quad f''(\rho_0)=\frac{p_0\gamma(\gamma-1)}{\rho_0^2}, \quad \dots \quad (4)$$

so that

$$\frac{\rho_0 f''(\rho_0)}{4a^3}+\frac{1}{2a}=\frac{\gamma+1}{4a}. \quad \dots \quad (5)$$

In the case of Boyle's law we have merely to make $\gamma=1$ in (5).

For ordinary gases $\gamma>1$ and the momentum is positive; but the above argument applies to all positive values of γ. If γ be negative, the pressure would increase as the density decreases, and the fluid would be essentially unstable.

However, a slightly modified form of (3) allows the exponent to be negative. If we take

$$p/p_0=2-(\rho/\rho_0)^{-\beta} \quad \dots \quad (6)$$

with β positive, we get as above

$$f'(\rho_0)=\frac{\beta p_0}{\rho_0}=a^2, \quad f''(\rho_0)=-\frac{(\beta+1)a^2}{\rho_0}, \quad \dots \quad (7)$$

and accordingly

$$\frac{\rho_0 f''(\rho_0)}{4a^3}+\frac{1}{2a}=\frac{1-\beta}{4a}. \quad \dots \quad (8)$$

* Communicated by the Author.

† Phil. Mag. vol. x. p. 364 (1905); Scientific Papers, vol. v. p. 265.

If $\beta=1$, the law of pressure is that under which waves can be propagated without a change of type, and we see that the momentum is zero. In general, the momentum is positive or negative according as β is less or greater than 1.

In the above formula (2) the calculation is approximate only, powers of the disturbance above the *second* being neglected. In the present note it is proposed to determine the sign of the momentum under the laws (3) and (6) more generally and further to extend the calculations to waves in a liquid moving in two dimensions under gravity.

It should be clearly understood that the discussion relates to *progressive* waves. If this restriction be dispensed with, it would always be possible to have a disturbance (limited if we please to a finite length) without momentum, as could be effected very simply by beginning with displacements unaccompanied by velocities. And the disturbance, considered as a whole, can never acquire (or lose) momentum. In order that a wave may be progressive in one direction only, a relation must subsist between the velocity and density at every point. In the case of Boyle's law this relation, first given by De Morgan *, is

$$u=a\log(\rho/\rho_0), \quad \dots \quad (9)$$

and more generally †

$$u=\int\sqrt{\left(\frac{dp}{d\rho}\right)}\cdot\frac{d\rho}{\rho} \quad \dots \quad (10)$$

Wherever this relation is violated, a wave emerges travelling in the negative direction.

For the adiabatic law (3), (10) gives

$$u=\frac{2a}{\gamma-1}\left\{\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}-1\right\}, \quad \dots \quad (11)$$

a being the velocity of infinitely small disturbances, and this reduces to (9) when $\gamma=1$. Whether γ be greater or less than 1, u is positive when ρ exceeds ρ_0. Similarly if the law of pressure be that expressed in (6),

$$u=\frac{2a}{\beta+1}\left\{1-\left(\frac{\rho}{\rho_0}\right)^{-\frac{\beta+1}{2}}\right\}. \quad \dots \quad (12)$$

Since β is positive, values of ρ greater than ρ_0 are here also accompanied by positive values of u.

* Airy, Phil. Mag. vol. xxxiv. p. 402 (1849).

† Earnshaw, Phil. Trans. 1859, p. 146.

By definition the momentum of the wave, whose length may be supposed to be limited, is per unit of cross-section

$$\int \rho u \, dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the integration extending over the whole length of the wave. If we introduce the value of u given in (11), we get

$$(13) = \frac{2\rho_0 a}{\gamma - 1} \int \left\{ \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma+1}{2}} - \frac{\rho}{\rho_0} \right\} dx \; ; \quad . \quad . \quad (14)$$

and the question to be examined is the sign of (14). For brevity we may write unity in place of ρ_0, and we suppose that the wave is such that its mean density is equal to that of the undisturbed fluid, so that $\int \rho \, dx = l$, where l is the length of the wave. If l be divided into n equal parts, then when n is great enough the integral may be represented by the sum

$$\left\{ \rho_1^{\frac{\gamma+1}{2}} + \rho_2^{\frac{\gamma+1}{2}} + \rho_3^{\frac{\gamma+1}{2}} + \ldots \quad - \rho_1 - \rho_2 - \ldots \right\} \frac{l}{n}, \quad (15)$$

in which all the ρ's are positive. Now it is a proposition in Algebra that

$$\frac{\rho_1^{\frac{\gamma+1}{2}} + \rho_2^{\frac{\gamma+1}{2}} + \ldots}{n} > \left(\frac{\rho_1 + \rho_2 + \ldots}{n} \right)^{\frac{\gamma+1}{2}}$$

when $\frac{1}{2}(\gamma+1)$ is negative, or positive and greater than unity; but that the reverse holds when $\frac{1}{2}(\gamma+1)$ is positive and less than unity. Of course the inequality becomes an equality when all the n quantities are equal. In the present application the sum of the ρ's is n, and under the adiabatic law (3), γ and $\frac{1}{2}(\gamma+1)$ are positive. Hence (15) is positive or negative according as $\frac{1}{2}(\gamma+1)$ is greater or less than unity, viz., according as γ is greater or less than unity. In either case the momentum represented by (13) is *positive*, and the conclusion is not limited to the supposition of small disturbances.

In like manner if the law of pressure be that expressed in (6), we get from (12)

$$(13) = \frac{2\rho_0 a}{\beta + 1} \int \left\{ \frac{\rho}{\rho_0} - \left(\frac{\rho}{\rho_0} \right)^{-\frac{\beta-1}{2}} \right\} dx, \quad . \quad . \quad (16)$$

from which we deduce almost exactly as before that the momentum (13) is positive if β (being positive) is less than 1

and negative if β is greater than 1. If $\beta = 1$, the momentum vanishes. The conclusions formerly obtained on the superposition of small disturbances are thus extended.

We will now discuss the momentum in certain cases of fluid motion under gravity. The simplest is that of *long* waves in a uniform canal. If η be the (small) elevation at any point x measured in the direction of the length of the canal and u the corresponding fluid velocity parallel to x, which is uniform over the section, the dynamical equation is*

$$\frac{du}{dt} = -g \frac{d\eta}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

As is well known, long waves of small elevation are propagated without change of form. If c be the velocity of propagation, a positive wave may be represented by

$$\eta = F(ct - x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where F denotes an arbitrary function, and c is related to the depth h_0 according to

$$c^2 = g h_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

From (17), (18)

$$u = \frac{g\eta}{c} = \sqrt{\left(\frac{g}{h_0} \right)} \cdot \eta \quad . \quad . \quad . \quad . \quad (20)$$

is the relation obtaining between the velocity and elevation at any place in a positive progressive wave of small elevation.

Equation (20), however, does not suffice for our present purpose. We may extend it by the consideration that in a long wave of finite disturbance the elevation and velocity may be taken as relative to the neighbouring parts of the wave. Thus, writing du for u and h for h_0, so that $\eta = dh$, we have

$$du = \sqrt{\left(\frac{g}{h} \right)} dh,$$

and on integration

$$u = 2\sqrt{g} \{ h^{\frac{1}{2}} + C \}.$$

The arbitrary constant of integration is determined by the fact that outside the wave $u = 0$ when $h = h_0$, whence and replacing h by $h_0 + \eta$, we get

$$u = 2\sqrt{g} \{ \sqrt{(h_0 + \eta)} - \sqrt{h_0} \}, \quad . \quad . \quad . \quad (21)$$

* Lamb's Hydrodynamics, § 168.

as the generalized form of (20). It is equivalent to a relation given first in another notation by De Morgan *, and it may be regarded as the condition which must be satisfied if the emergence of a negative wave is to be obviated.

We are now prepared to calculate the momentum. For a wave in which the mean elevation is zero, the momentum corresponding to unit horizontal breadth is

$$\rho\int u(h_0+\eta)dx=\tfrac{3}{4}\rho\sqrt{(g/h_0)}\int\eta^2dx, \quad . \quad . \quad (22)$$

when we omit cubes and higher powers of η. We may write (22) also in the form

$$\text{Momentum}=\frac{3}{4}\frac{\text{Total Energy}}{c}, \quad . \quad . \quad . \quad . \quad (23)$$

c being the velocity of propagation of waves of small elevation.

As in (14), with γ equal to 2, we may prove that the momentum is positive without restriction upon the value of η.

As another example, periodic waves moving on the surface of deep water may also be referred to. The momentum of such waves has been calculated by Lamb †, on the basis of Stokes' second approximation. It appears that the momentum per wave-length and per unit width perpendicular to the plane of motion is

$$\pi\rho a^2c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where c is the velocity of propagation of the waves in question and the wave form is approximately

$$\eta=a\cos\frac{2\pi}{\lambda}(ct-x) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The forward velocity of the surface layers was remarked by Stokes. For a simple view of the matter reference may be made also to Phil. Mag. vol. i. p. 257 (1876) ; Scientific Papers, vol. i. p. 263.

* Airy, Phil. Mag. vol. xxxiv. p. 402 (1849).
† Hydrodynamics, § 246.

XLVIII. *The Expression for the Electrical Conductivity of Metals as deduced from the Electron Theory.* By W. F. G. Swann, *D.Sc., A.R.C.S., Assistant Lecturer in Physics at the University of Sheffield* *.

Introduction.

THE theory of the electrical conductivity of metals has been worked out on many assumptions. One of the simplest and best known of these methods is that employed by Drude, in which the assumption is made that in the absence of the electric field all the electrons move with the same velocity, and that the velocity produced by the field in an electron is the velocity which is produced in it while it is travelling between two points of collision, the essential assumption being that at each collision the effect of all previous actions of the field on the electron are wiped out. The value of the conductivity σ which has been deduced from these assumptions is

$$\sigma=\frac{ne^2\lambda v}{4\alpha\theta}\,\dagger, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where n is the number of electrons per c.c., λ is the mean free path, v is the velocity, and $\alpha\theta$ is the kinetic energy of a gas molecule at a temperature θ.

The object of Part I. of the present paper is to show that the above assumptions do not lead to (1), but to the formula

$$\sigma=\frac{ne^2\lambda v}{3\alpha\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The difference between (1) and (2) is partly due to what is, in the opinion of the author, an improper use of the quantity known as the mean free path, and partly due to another cause which will be better understood at a later stage of the paper.

The thermal conductivity k calculated with the proper use of the mean free path gives, for the above case, the ordinarily accepted value $k=\frac{1}{3}n\lambda v\alpha$; and the interesting point is, that while at 0° C. (1) gives $k/\sigma=6{\cdot}3\times10^{10}$, (2) results in $k/\sigma=4{\cdot}7\times10^{10}$. The experimentally found value of k/σ for most pure metals is about $6{\cdot}3\times10^{10}$ at 0° C.; so that the conclusion to be drawn is, that the assumptions on which (1) and (2) are based are nothing like as representative of the

* Communicated by the Author (now of the Carnegie Institution of Washington).
† J. J. Thomson, 'Corpuscular Theory of Matter,' p. 56.