The mean of these measurements is equal to 21·0609, exactly agreeing with the value found by Fabry and Perot.

It is to be remarked that we have here used only for convenience sake one of the yellow lines of mercury as a reference line; evidently the difference in wave-lengths can be easily referred to standard lines of cadmium. We believe that the present method can be applied to various other purposes of accurate spectroscopic work.

August 7, 1913.

XIII. *The Classification of Electromagnetic Fields.* By H. Bateman, *M.A., Ph.D., Johnston Research Scholar, Johns Hopkins University* *.

§ 1. THE field of the two vectors E, H will be called electromagnetic when Maxwell's equations

$$\left.\begin{array}{ll}\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=\frac{\partial E_x}{\partial t}, & \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial H_x}{\partial t}, \\ \cdots & \cdots \\ \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0, & \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0,\end{array}\right\} \quad (1)$$

are satisfied for some real domain of the variables x, y, z, t. In general these equations cannot be regarded as holding for all real values of x, y, z, t, and the exceptional space-time points or domains of such points are to be regarded as the singularities of the electromagnetic field †.

The simplest type of point singularity is one which moves with a velocity less than unity, along an arbitrary curve. If we solve equations (1) in the usual way with the aid of four potentials A_x, A_y, A_z, Φ satisfying the wave-equation

$$\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}+\frac{\partial^2\Omega}{\partial z^2}=\frac{\partial^2\Omega}{\partial t^2}, \quad . \quad . \quad . \quad . \quad (2)$$

and the relations

$$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{\partial \Phi}{\partial t}=0, \quad . \quad . \quad . \quad (3)$$

$$\begin{array}{ll}\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=H_x, & \frac{\partial A_x}{\partial t}+\frac{\partial \Phi}{\partial x}=-E_x; \\ \cdots & \cdots\end{array} \quad (4)$$

* Communicated by the Author.

† We shall suppose that E_x, H_x, &c., are one-valued functions of x, y, z, t for the whole of the real domain of these variables, the point singularities being excluded.

an electromagnetic field with this simple type of singularity is obtained by putting

$$A_x=\frac{\xi'(\tau)}{M}, \quad A_y=\frac{\eta'(\tau)}{M}, \quad A_z=\frac{\rho'(\tau)}{M}, \quad \Phi=\frac{1}{M}, \quad (5)$$

where τ, M are defined by the equations

$$[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2=(t-\tau)^2, \quad t \geqslant \tau, \quad (6)$$

$$\xi'^2(\tau)+\eta'^2(\tau)+\zeta'^2(\tau)<1, \quad . \quad . \quad . \quad (7)$$

$$M=\xi'(\tau)[x-\xi]+\eta'(\tau)[y-\eta]+\zeta'(\tau)[z-\zeta]-(t-\tau). \quad (8)$$

If all these conditions are satisfied, there is only one value of τ for each point (x, y, z, t), and M only vanishes when all the terms in (6) are zero, *i. e.* when x, y, z, t coincides with the moving-point (ξ, η, ζ) *. This electromagnetic field, which was discovered by Liénard, is rightly regarded as being of fundamental importance in the electron theory of matter, and electromagnetic fields which can be obtained by superposing a number of fields of this simple type are studied almost exclusively.

For the sake of thoroughness, however, it is desirable that all types of electromagnetic fields should be studied, the aim being, if possible, to discover a number of fundamental types from which all real electromagnetic fields can be derived by superposition †.

§ 2. Solutions analogous to Liénard's may be obtained by discarding the inequality (7), or by considering complex functions ξ, η, ζ and a complex variable τ, or, finally, by putting

$$A_x=\Sigma\pm\frac{\xi'(u)}{M}, \quad A_y=\Sigma\pm\frac{\eta'(u)}{M}, \quad A_z=\Sigma\pm\frac{\zeta'(u)}{M}, \quad \Phi=0, \quad (9)$$

where the summation extends over values of u for which ‡

$$[x-\xi(u)]^2+[y-\eta(u)]^2+[z-\zeta(u)]^2=t^2, \quad . \quad (10)$$

and $$M=\xi'(u)[x-\xi]+\eta'(u)[y-\eta]+\zeta'(u)[z-\zeta]. \quad (11)$$

In the first case the singularity (ξ, η, ζ, τ) travels with a

* For further details, see Schott's 'Electromagnetic Radiation.

† Differentiations with regard to the variables x, y, z, t are supposed to be included, as well as integrations with regard to variable parameters.

‡ Any finite number of roots may be chosen and the signs in (9) distributed arbitrarily and Maxwell's equations will be satisfied. In making a choice of the roots and signs we must endeavour to make the components of E and H single-valued functions of x, y, z, t.

velocity greater than or equal to that of light. This case has been discussed at some length by Sommerfeld and Schott (*l. c.*). In the case of the electromagnetic field defined by the potentials (9) there are primary singularities distributed along the curve $x=\xi(u)$, $y=\eta(u)$, $z=\zeta(u)$ at time $t=0$, and these give rise to secondary singularities which at time t lie on a tubular surface having the given curve as axis. If we write $x-a$, $y-b$, $z-c$, $t-\tau$ in place of x, y, z, t, and integrate over a suitable domain of the variables a, b, c, τ, we may get rid of the awkward infinities of the electric and magnetic vectors, but the derivatives of these vectors will not all be continuous over the regions occupied by these infinities. The chief peculiarity of an electromagnetic field of this kind is that a portion of matter sends out radiations for a finite interval of time, and the radiations, which seem to be partly of a material nature, travel outwards with the velocity of light. The radiated matter, however, seems generally to fill the spaces between a number of pairs of moving surfaces, and so the present type of electromagnetic field is essentially different from any of the fields which have so far been observed in nature.

An electromagnetic field of a more promising nature is obtained by writing

$$A_x=\Sigma\frac{l(u)}{w},\quad A_y=\Sigma\frac{m(u)}{w},\quad A_z=\Sigma\frac{n(u)}{w},\quad \Phi=\Sigma\frac{1}{w},\quad . \quad (12)$$

where the summation extends over some of the values of u for which equation (10) is satisfied and

$$l^2+m^2+n^2=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$l\xi'+m\eta'+n\zeta'=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$w=l(x-\xi)+m(y-\eta)+n(z-\zeta)-t. \quad . \quad . \quad . \quad (15)$$

If l, m, n are real, the quantity w vanishes only when

$$\frac{x-\xi}{l}=\frac{y-\eta}{m}=\frac{z-\zeta}{n}=\frac{t}{1}. \quad . \quad . \quad . \quad . \quad (16)$$

The singularities of the electromagnetic field may be described by saying that there are guns distributed along the curve

$$x=\xi(u),\quad y=\eta(u),\quad z=\zeta(u). \quad . \quad . \quad . \quad . \quad (17)$$

Each gun points in a direction at right angles to the curve, and fires out a singularity or bullet at time $t=0$. The bullets all move in straight lines with the velocity of light.

This type of electromagnetic field may be generalized by integration just like the previous one. In the resulting electromagnetic field material particles are fired out from a material wire for a certain interval of time, and these particles move along straight lines with the velocity of light. The way in which the shape and size of such a particle varies during its motion has not yet been ascertained.

§ 3. We shall now discuss another class of electromagnetic fields in which the radiated energy is concentrated round certain moving points which travel with the speed of light, but in this case the difficulty with regard to the roots of (10) is absent.

Defining τ as before by means of equations (6) and (7), we choose 16 functions of τ which satisfy the equations

$$\left.\begin{array}{lll} l^2+m^2+n^2=p^2, & l_0^2+m_0^2+n_0^2=p_0^2, & \lambda^2+\mu^2+\nu^2=\varpi^2, \\ & \lambda_0^2+\mu_0^2+\nu_0^2=\varpi_0^2, & \end{array}\right\} \quad (18)$$

$$\left.\begin{array}{lll} l\lambda+m\mu+n\nu=p\varpi, & l_0\lambda+m_0\mu+n_0\nu=p_0\varpi, & l\lambda_0+m\mu_0+n\nu_0=p\varpi_0, \\ & l_0\lambda_0+m_0\mu_0+n_0\nu_0=p_0\varpi_0\,; & \end{array}\right\} \quad (19)$$

and write

$$\left.\begin{aligned} w &= l(x-\xi)+m(y-\eta)+n(z-\zeta)-p(t-\tau), \\ w_0 &= l_0(x-\xi)+m_0(y-\eta)+n_0(z-\zeta)-p_0(t-\tau), \\ \sigma &= \lambda(x-\xi)+\mu(y-\eta)+\nu(z-\zeta)-\varpi(t-\tau), \\ \sigma_0 &= \lambda_0(x-\xi)+\mu_0(y-\eta)+\nu_0(z-\zeta)-\varpi_0(t-\tau). \end{aligned}\right\} \quad (20)$$

If M is defined by equation (8), it is easy to prove that when f and F are arbitrary functions the expressions

$$\Omega=\frac{1}{M}f\left(\frac{\sigma}{w},\tau\right),\qquad \Omega_0=\frac{1}{M}F\left(\frac{\sigma}{w_0},\tau\right) \quad . \quad (21)$$

satisfy the wave-equation (2), and that the potentials

$$\left.\begin{array}{ll} A_x=\dfrac{l}{M}\dfrac{\sigma}{w}+\dfrac{l_0}{M}\dfrac{\sigma}{w_0}-\dfrac{2\lambda}{M}, & A_y=\dfrac{m}{M}\dfrac{\sigma}{w}+\dfrac{m_0}{M}\dfrac{\sigma}{w_0}-\dfrac{2\mu}{M}, \\ A_z=\dfrac{n}{M}\dfrac{\sigma}{w}+\dfrac{n_0}{M}\dfrac{\sigma}{w_0}-\dfrac{2\nu}{M}, & \Phi=\dfrac{p}{M}\dfrac{\sigma}{w}+\dfrac{p_0}{M}\dfrac{\sigma}{w_0}-\dfrac{2\varpi}{M}, \end{array}\right\} \quad (22)$$

which consequently satisfy (2), are connected by the relation (3).

The electromagnetic field specified by these potentials has singularities at space-time points for which σ_0 is zero. To see this we remark that there is a relation of the form

$$\sigma\sigma_0=\theta w w_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where θ is some function of τ. The existence of a relation of this type is easily realized by considering the particular case when $p_0=p$, $\varpi_0=\varpi$, and

$$\left.\begin{array}{lll} \lambda=\dfrac{p\varpi}{q}, & \mu=0, & \nu=\dfrac{\varpi}{q}\sqrt{q^2-p^2}\,; \\ \lambda_0=\dfrac{p\varpi}{q}, & \mu_0=0, & \nu_0=-\dfrac{\varpi}{q}\sqrt{q^2-p^2}\,; \\ l=q, & m=\sqrt{p^2-q^2}, & n=0\,; \\ l_0=q, & m_0=-\sqrt{p^2-q^2}, & n_0=0. \end{array}\right\} \quad (24)$$

The relation (23) is then verified at once by using (6).

Since the general values of λ, μ, ν, &c., may be derived from these particular ones by a suitable orthogonal substitution, it is easy to see that a relation of type (23) holds universally.

We shall now assume that $\lambda, \mu, \nu, \varpi, \lambda_0, \mu_0, \nu_0, \varpi_0$ are all real, then it is easily seen that σ_0 is zero when

$$\frac{x-\xi}{\lambda_0}=\frac{y-\eta}{\mu_0}=\frac{z-\zeta}{\nu_0}=\frac{t-\tau}{\varpi_0}.$$

Hence the electromagnetic field has a singularity which starts from the point (ξ, η, ζ, τ) and moves with the speed of light along a straight line whose direction cosines are proportional to $(\lambda_0, \mu_0, \nu_0)$. Taking each point (ξ, η, ζ, τ) in turn, we obtain all the singularities of the field in this way. The moving point (ξ, η, ζ, τ) may be described as a *gun* which moves about in an arbitrary prescribed manner and fires out "bullets" which travel with the speed of light. The direction in which the gun points at any instant can be chosen arbitrarily provided the differential coefficients of the functions λ, μ, ν ... are continuous.

After a long calculation we find that the component of the electric force along the radius from (ξ, η, ζ, τ) to the point (x, y, z, t) at which the force is required is

$$\frac{2}{M^2}(\lambda\xi'+\mu\eta'+\nu\zeta'-\varpi)-\frac{\sigma\rho}{M^2 w}-\frac{\sigma\rho_0}{M^2 w_0},$$

where

$$\rho\ =l\xi'+m\eta'+n\zeta'-p,$$
$$\rho_0=l_0\xi'+m_0\eta'+n_0\zeta'-p_0.$$

We shall assume that ρ and ρ_0 vanish so that our expression may be finite for $\sigma_0=0$.

Integrating over the surface of a sphere for which $t-\tau$ is constant, we find that the total charge associated with the singularity (ξ, η, ζ, τ) is

$$8\pi\,.\,\frac{\lambda\xi'+\mu\eta'+\nu\zeta'-\varpi}{1-\xi'^2-\eta'^2-\zeta'^2}.$$

In general the charge varies with τ: to make it constant we must introduce a restriction such as

$$\lambda\xi'+\mu\eta'+\nu\zeta'-\varpi=\xi'^2+\eta'^2+\zeta'^2-1. \quad . \quad . \quad (25)$$

If, in particular, we take $\varpi=1$, this condition implies that the angle between the direction of the gun and the direction of the gun's motion has a cosine equal to v, where v is the velocity of the gun. We may get rid of the charge at the singularity (ξ, η, ζ, τ) by adding suitable multiples of Liénard's potentials (5). To simplify our expressions, we shall add to (19) the additional equations

$$\varpi=\varpi_0=p=p_0=1, \quad l\xi'+m\eta'+n\zeta'=1, \quad l_0\xi'+m_0\eta'+n_0\zeta'=1, \quad (26)$$

which, together with (19), lead to (25). The electromagnetic field specified by the potentials

$$\left.\begin{aligned} A_x&=\frac{l}{2M}\frac{\sigma}{w}+\frac{l_0}{2M}\frac{\sigma}{w_0}-\frac{\lambda-\xi'}{M},\\ A_y&=\frac{m}{2M}\frac{\sigma}{w}+\frac{m_0}{2M}\frac{\sigma}{w_0}-\frac{\mu-\eta'}{M},\\ A_z&=\frac{n}{2M}\frac{\sigma}{w}+\frac{n_0}{2M}\frac{\sigma}{w_0}-\frac{\nu-\zeta'}{M},\\ \Phi&=\frac{1}{2M}\left(\frac{\sigma}{w}+\frac{\sigma}{w_0}\right), \end{aligned}\right\} \quad . \quad . \quad (27)$$

then has no charge associated with the singularity (ξ, η, ζ, τ), and both the electric and magnetic forces at (x, y, z, t) are perpendicular to the radius from (ξ, η, ζ, τ).

If we add to (27) the field specified by the potentials

$$\left.\begin{aligned} A_x'&=\frac{l}{2M}\frac{\sigma_0}{w}+\frac{l_0\sigma_0}{2Mw_0}-\frac{\lambda_0-\xi'}{M},\\ A_y'&=\frac{m}{2M}\frac{\sigma_0}{w}+\frac{m_0}{2M}\frac{\sigma_0}{w_0}-\frac{\mu_0-\eta'}{M},\\ A_z'&=\frac{n}{2M}\frac{\sigma_0}{w}+\frac{n_0}{2M}\frac{\sigma_0}{w_0}-\frac{\nu_0-\zeta'}{M}\\ \Phi'&=\frac{1}{2M}\left(\frac{\sigma_0}{w}+\frac{\sigma_0}{w_0}\right), \end{aligned}\right\} \quad . \quad (28)$$

and make use of the relations

$$\lambda+\lambda_0=2\xi', \quad \mu+\mu_0=2\eta', \quad \nu+\nu_0=2\zeta', \quad \sigma+\sigma_0=2\mathrm{M}, \quad (29)$$

we obtain the field specified by the potentials

$$\left.\begin{aligned} a_x &= \frac{l}{w}+\frac{l_0}{w_0}, & a_y &= \frac{m}{w}+\frac{m_0}{w_0}, \\ a_z &= \frac{n}{w}+\frac{n_0}{w_0}, & \phi &= \frac{1}{w}+\frac{1}{w_0}. \end{aligned}\right\} \quad (30)$$

The properties of this type of electromagnetic field have been discussed in a previous paper *.

When w, w_0, σ, σ_0 are defined by equations (19) the electromagnetic field specified by the potentials

$$\left.\begin{aligned} i\mathrm{A}_x &= \frac{l}{\mathrm{M}}\frac{\sigma}{w}-\frac{l_0}{\mathrm{M}}\frac{\sigma}{w_0}, & i\mathrm{A}_y &= \frac{m}{\mathrm{M}}\frac{\sigma}{w}-\frac{m_0}{\mathrm{M}}\frac{\sigma}{w_0}, \\ i\mathrm{A}_z &= \frac{n}{\mathrm{M}}\frac{\sigma}{w}-\frac{n_0}{\mathrm{M}}\frac{\sigma}{w_0}, & i\Phi &= \frac{p}{\mathrm{M}}\frac{\sigma}{w}-\frac{p_0}{\mathrm{M}}\frac{\sigma}{w_0}, \end{aligned}\right\} . \quad (31)$$

has singularities at points for which σ_0 is zero and the electric charge associated with the point (ξ, η, ζ, τ) is zero. If in addition equations (26) are satisfied and we add to the potentials (31), the corresponding ones in which σ is replaced by σ_0, we obtain the potentials

$$\left.\begin{aligned} ia_x &= \frac{l}{w}-\frac{l_0}{w_0}, \quad ia_y = \frac{m}{w}-\frac{m_0}{w_0}, \quad ia_z = \frac{n}{w}-\frac{n_0}{w_0}, \\ \phi &= \frac{1}{w}-\frac{1}{w_0}, \end{aligned}\right\} \quad (32)$$

These specify an electromagnetic field in which both the electric and magnetic forces at (x, y, z, t) are perpendicular to the radius from ξ, η, ζ, τ. The field has singularities at points where w and w_0, and consequently σ or σ_0, are zero. The moving point ξ, η, ζ, τ may be described as a gun which fires out magnetic doublets in directions for which σ and σ_0 vanish respectively. If we write

$$\frac{x-\xi}{t-\tau}=\alpha, \qquad \frac{y-\eta}{t-\tau}=\beta, \qquad \frac{z-\zeta}{t-\tau}=\gamma,$$

and treat α, β, γ as constants, the components of the electric

* Phil. Mag. Oct. 1913.

and magnetic vectors may be expressed in the forms

$$\left.\begin{aligned} i\mathrm{E}_x &= \frac{1}{\mathrm{M}}\frac{d}{d\tau}\left[\frac{l-\alpha}{l\alpha+m\beta+n\gamma-1}-\frac{l_0-\alpha}{l_0\alpha+m_0\beta+n_0\gamma-1}\right], \\ &= \frac{i}{\mathrm{M}}\frac{d}{d\tau}\left[\frac{\nu\beta-\mu\gamma}{\lambda\alpha+\mu\beta+\nu\gamma-1}-\frac{\nu_0\beta-\mu_0\gamma}{\lambda_0\alpha+\mu_0\beta+\nu_0\gamma-1}\right], \\ i\mathrm{H}_x &= \frac{1}{\mathrm{M}}\frac{d}{d\tau}\left[\frac{\beta n-\gamma m}{l\alpha+m\beta+n\gamma-1}-\frac{\beta n_0-\gamma m_0}{l_0\alpha+m_0\beta+n_0\gamma-1}\right], \\ &= -\frac{i}{\mathrm{M}}\frac{d}{d\tau}\left[\frac{\lambda-\alpha}{\lambda\alpha+\mu\beta+\nu\gamma-1}-\frac{\lambda_0-\alpha}{\lambda_0\alpha+\mu_0\beta+\nu_0\gamma-1}\right]. \end{aligned}\right\} \quad (33)$$

With the aid of these expressions we can show that the lines of electric and magnetic force on a sphere whose centre is at (ξ, η, ζ, τ) are such that when the sphere is inverted into a plane the electric lines of force are represented by the equipotential lines due to two doublets, and the magnetic lines of force by the corresponding stream lines, the flow being in two dimensions. The bullets are thus magnetic doublets.

§4. A very general type of electromagnetic field in which the electric and magnetic forces at x, y, z, t are perpendicular to the radius from ξ, η, ζ, τ may be obtained as follows:—

Let w, w_0, σ be defined by equations (20) and write

$$\theta = f\left(\frac{\sigma}{w}, \tau\right)+f\left(\frac{\sigma}{w_0}, \tau\right),$$

$$\left.\begin{aligned} \mathrm{H}_x &= \frac{\partial(\theta, \tau)}{\partial(y, z)}, & \mathrm{H}_y &= \frac{\partial(\theta, \tau)}{\partial(z, x)}, & \mathrm{H}_z &= \frac{\partial(\theta, \tau)}{\partial(x, y)}; \\ \mathrm{E}_x &= \frac{\partial(\theta, \tau)}{\partial(x, t)}, & \mathrm{E}_y &= \frac{\partial(\theta, \tau)}{\partial(y, t)}, & \mathrm{E}_z &= \frac{\partial(\theta, \tau)}{\partial(z, t)}. \end{aligned}\right\} \quad (34)$$

Then, if

$$i\nu = f\left(\frac{\sigma}{w}, \tau\right)-f\left(\frac{\sigma}{w_0}, \tau\right),$$

we have also

$$\left.\begin{aligned} \mathrm{H}_x &= -\frac{\partial(\nu, \tau)}{\partial(x, t)}, & \mathrm{H}_y &= -\frac{\partial(\nu, \tau)}{\partial(y, t)}, & \mathrm{H}_z &= -\frac{\partial(\nu, \tau)}{\partial(z, t)}; \\ \mathrm{E}_x &= \frac{\partial(\nu, \tau)}{\partial(y, z)}, & \mathrm{E}_y &= \frac{\partial(\nu, \tau)}{\partial(z, x)}, & \mathrm{E}_z &= \frac{\partial(\nu, \tau)}{\partial(x, y)}; \end{aligned}\right\} \quad (35)$$

and it is evident from these two sets of relations that Maxwell's equations (1) are satisfied.

To verify these identities we have to show that if

$$s=\theta+i\nu=2f\left(\frac{\sigma}{w},\ \tau\right),$$

$$\frac{\partial(s,\tau)}{\partial(y,z)}=i\frac{\partial(s,\tau)}{\partial(x,t)},\quad \frac{\partial(s,\tau)}{\partial(z,x)}=i\frac{\partial(s,\tau)}{\partial(y,t)},\quad \frac{\partial(s,\tau)}{\partial(x,y)}=i\frac{\partial(s,\tau)}{\partial(z,t)}. \quad (36)$$

Now we find at once that

$$\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2+\left(\frac{\partial s}{\partial z}\right)^2=\left(\frac{\partial s}{\partial t}\right)^2,$$

$$\frac{\partial s}{\partial x}\frac{\partial \tau}{\partial x}+\frac{\partial s}{\partial y}\frac{\partial \tau}{\partial y}+\frac{\partial s}{\partial z}\frac{\partial \tau}{\partial z}=\frac{\partial s}{\partial t}\frac{\partial \tau}{\partial t},$$

$$\left(\frac{\partial \tau}{\partial x}\right)^2+\left(\frac{\partial \tau}{\partial y}\right)^2+\left(\frac{\partial \tau}{\partial z}\right)^2=\left(\frac{\partial \tau}{\partial t}\right)^2,$$

$$\frac{\partial s}{\partial x}\cdot\frac{\partial(s,\tau)}{\partial(x,t)}+\frac{\partial s}{\partial y}\cdot\frac{\partial(s,\tau)}{\partial(y,t)}+\frac{\partial s}{\partial z}\cdot\frac{\partial(s,\tau)}{\partial(z,t)}=0,$$

$$\frac{\partial \tau}{\partial x}\cdot\frac{\partial(s,\tau)}{\partial(x,t)}+\frac{\partial \tau}{\partial y}\cdot\frac{\partial(s,\tau)}{\partial(y,t)}+\frac{\partial \tau}{\partial z}\cdot\frac{\partial(s,\tau)}{\partial(z,t)}=0.$$

Hence it will be sufficient to verify the first of the equations (36). To do this we must show that

$$\left(\mu\frac{\partial\tau}{\partial z}-\nu\frac{\partial\tau}{\partial y}-i\lambda\frac{\partial\tau}{\partial t}-i\varpi\frac{\partial\tau}{\partial x}\right)w=\left(m\frac{\partial\tau}{\partial z}-n\frac{\partial\tau}{\partial y}-il\frac{\partial\tau}{\partial t}-ip\frac{\partial\tau}{\partial x}\right).$$

If now we use the values (24) this relation is easily seen to be satisfied in virtue of (6). We may infer, then, that it is satisfied in the general case.

In the electromagnetic field (34) the electric and magnetic forces at (x, y, z, t) are at right angles to the radius from (ξ, η, ζ, τ); they are also at right angles to one another and equal in magnitude. The moving point (ξ, η, ζ, τ) again has the character of a gun which fires out bullets which move with the speed of light and are singularities of the electromagnetic field.

The lines of electric and magnetic force on a sphere whose centre is the point (ξ, η, ζ, τ) are easily drawn. It follows at once from (34) and (35) that the magnetic lines of force are given by θ=constant and the electric lines of force by ν=constant. By choosing the function $f\left(\frac{\sigma}{w},\ \tau\right)$ in a suitable way we can make the distribution of the lines of force satisfy certain prescribed conditions.

§5. The electromagnetic fields of sections 3 and 4 may evidently be generalized by writing $x+a$, $y+b$, $z+c$, $t+\epsilon$ in place of x, y, z, t and integrating over some domain of the point (a, b, c, ϵ). It should be possible in this way to get rid of the point singularities and replace them by material particles throughout which the electric and magnetic forces are finite, but do not necessarily satisfy Maxwell's equations.

The electromagnetic fields of sections 3 and 4 may also be generalized by adopting the method of section 2, but the analysis is not of sufficient interest to be given in detail.

The discussion of the singularities which has been given here is not exhaustive, for we have omitted the case of moving singularities at infinity; the fields which are obtained in this way, however, may be regarded as limiting cases of those which have already been discussed. The case of a primary singularity which moves with a velocity greater than that of light has been mentioned only very briefly. The investigation given in Schott's 'Electromagnetic Radiation' will give an idea of what is to be expected in this case.

Summing up the results of our investigations we are able to enumerate four distinct types of elementary electromagnetic fields.

In a field of the first type there is one point singularity which moves with a velocity less than that of light, and a constant electric charge is associated with the singularity. An electromagnetic field which can be obtained by superposing elementary fields of the first type will be said to belong to class A. We shall include under class A′ fields which can be obtained by superposing the fields due to Hertzian oscillators in motion, an elementary field of this type being represented by the potentials *

$$\mathrm{A}=\operatorname{curl}\Gamma-\frac{\partial\Omega}{\partial t}+\operatorname{grad}\Psi,\qquad \Phi=\operatorname{div}\Omega-\frac{\partial\Psi}{\partial t}, \quad (37)$$

where the vectors Γ, Ω and the scalar Ψ are functions of the type $\frac{f(\tau)}{\mathrm{M}}$, τ and M having the same meaning as in §1. It is probable that all fields of class A′ can be regarded as belonging to class A.

In a field of the second type there is a point singularity (the gun) which moves with a velocity less than that of light, and point singularities are fired out from the gun with

* These are immediate generalizations of the potentials used by Prof. E. T. Whittaker, Proc. London Math. Soc. ser. 2, vol. i. (1903).

velocities equal to that of light, an electromagnetic field which is built up from elementary fields of this type will be said to belong to class B.

In a field of the third type the primary singularities lie along a curve at some instant τ, at any subsequent time there are singularities distributed all over a tubular surface having the curve as axis. An electromagnetic field which is built up from elementary fields of this type will be said to belong to class C.

In a field of the fourth type the primary singularity moves with a velocity which is sometimes (or always) greater than that of light, and secondary singularities are fired out in various directions with the velocity of light. An electromagnetic field which is built up from elementary fields of this type will be said to belong to class D.

It is possible that all these different types of electromagnetic fields can be obtained from a single fundamental type by superposition. The fundamental solution of the equation of wave-motion which seems to be the most suitable for such a purpose is of the form

$$\Omega=\frac{1}{(x-a)^2+(y-b)^2+(z-c)^2-(t-\epsilon)^2}$$
$$\times f\left[\frac{\lambda(x-a)+\mu(y-b)+\nu(z-c)-\varpi(t-\epsilon)}{l(x-a)+m(y-b)+n(z-c)-p(t-\epsilon)}\right], \quad (38)$$

where $a, b, c, \epsilon, l, m, n, p, \lambda, \mu, \nu, \varpi$ are constants satisfying equations (18) and (19). To obtain from this a solution of the form (21) we must either regard a, b, c as functions of the complex variable ϵ and integrate round a closed contour in the complex plane *, or else regard a, b, c as functions of the real variable ϵ and take the principal value of an integral. The difficulties of the analysis are, however, so great that it seems better to retain the four different classes of fields with the understanding that it may be necessary to supplement them.

When the list is complete any real field which can be derived by superposition from potentials of type (38) and which is such that the electric and magnetic forces never become infinite (although they may not satisfy Maxwell's equations over the whole domain of the real variables x, y, z, t), ought to be obtainable by superposition from fields belonging to the different classes. A question which will then have to be settled is that of the uniqueness of the representation. It seems likely that there is only one set of elementary fields which, when added together, will be equivalent to a given electromagnetic field for the whole domain of the real variables x, y, z, t, but this proposition evidently requires proof.

A further difficulty arises from the fact that we can find wave-functions which are homogeneous functions of degree n in $x-a$, $y-b$, $z-c$, $t-\epsilon$, where n is not an integer. There are also many valued wave-functions such as $\tan^{-1}\frac{y}{x}$. Electromagnetic fields in which the components of E and H are represented by wave-functions of this type are, however, excluded by the condition, stated at the outset, that the components of E and H must be one-valued functions of x, y, z, t for the whole real domain of these variables.

* This method has been used by Prof. A. W. Conway to obtain Liénard's potentials. Proc. London Math. Soc. ser. 2, vol. i.

XIV. *On the Action of a disturbing Force in the restricted Problem of Three Bodies.* By R. J. Pocock, *B.A., B.Sc., Queen's College, Oxford* *.

In the restricted problem of three bodies, let H, J represent the two larger bodies, P the particle. Let J be of mass unity, H of mass ν, and take HJ as unit distance. Let n be the mean angular velocity of J; r, ρ the bipolar co-ordinates of P referred to H and J. Then with reference to HJ and a perpendicular thereto through the C. G. of H & J as moving axes rotating with angular velocity n, we have the equations

$$\left.\begin{aligned}\ddot{x}-2.n.\dot{y}&=\frac{\partial\Omega}{\partial x}\\ \ddot{y}+2.n.\dot{x}&=\frac{\partial\Omega}{\partial y}\end{aligned}\right\}\qquad n^2=\nu+1,$$

$$2\Omega=\nu\left(r^2+\frac{2}{r}\right)+\rho^2+\frac{2}{\rho};\quad r^2=x^2+y^2;\quad \rho^2=(x-1)^2+y^2.$$

These equations admit of Jacobi's integral, viz.:

$$V^2=\dot{x}^2+\dot{y}^2=2\Omega-C,$$

where V denotes the velocity of P and C is a constant.

2. Let us now suppose that an additional body S of mass μ is introduced into the system. It is assumed that S is small

* Communicated by the Author.