Chapter 11

Discussion of Vorton Results

In this chapter, the results obtained from the numerical simulations presented in Chapter 10 will be discussed against the background of the requirements we have posed in §7.2 on vortex methods. In §11.2 we separately discuss the simulation of §10.6 (a single vorton ring in a shear flow over a flat plate) and its relevance for research on coherent structures (CS). Finally, in §11.3, conclusions are summarized regarding the applicability of the vorton method which has been used in this thesis. Besides, suggestions are made for possible improvements of the method.

While reading the discussion presented below, the reader has to bear in mind that we have concentrated on two main questions:

- Is it possible to obtain an adequate [1] representation of vortex structures by means of vortons (*in casu* vorton rings representing vortex rings)?
- Can the deformation and interaction of vorton rings be regarded as an adequate representation of physical vortex ring phenomena [2]?

For convenience, we will refer to our vorton simulations by means of the numbers of the sections in which they have been treated:

- §10.1 = single vorton ring
- §10.2 = single pseudo-elliptical vorton ring
- §10.3 = two head-on colliding vorton rings
- §10.4 = two obliquely interacting vorton rings
- §10.5 = two knotted vorton rings

11.1 Satisfaction of Vortex Method Requirements

11.1.1 Divergence-free Vorticity Field

As discussed in §9.2, this requirement on vortex methods is fulfilled due to our derivation of the vorton vorticity field. As a consequence we have been able to remove the inconsistency between the N- and K-equation, as described in §9.3.

We have compared simulations for which the N-equation and the K-equation have been applied with those for which the N+K-equation has been applied. From the simulations in §10.2 (fig.10.8), §10.4 (fig.10.32), and §10.5, we conclude that application of the K-equations

[1] By "adequate" we mean a representation which at least shows qualitative characteristics similar to those which have been found by experimental investigation.

[2] These phenomena have been mentioned in §9.1.

gives unreliable results [3]. From the simulations in §10.4 (fig.10.32 and fig.10.33) and the results mentioned in §10.5, we conclude that the N+K-equation is to be preferred above the N-equation, i.e. the equation originally proposed by Novikov [168].

11.1.2 Correct Modelling of Continuous Distributions of Vorticity

Regarding the representation of a vortex ring by means of a vorton ring, we observe that a core can be attributed to the vorton ring, in which the distribution of vorticity agrees qualitatively with the (scarce) experimental results on this issue (see fig.10.2). At first glance, this result may seem surprising, since vortons are generally regarded as 3-D point-vortices with zero core size [4].

The core prevents the vortons from approaching each other to arbitrarily close distances (see §10.3, fig.10.16). This result weakens one of the arguments against the vorton method, i.e. its failure when vortons approach to small distances.

The core size, however, depends on the distance between the vortons in a ring (see fig.10.3). This means that the number of vortons is restricted by the quantitative characteristics (i.e. its velocity, radius, circulation) of the ring that we want to simulate. In §10.4, we have seen that this may lead to a rather small number of vortons in a ring and to a clash with the requirements of numerical accuracy which we would like to impose. However, our simulations have shown that even for such small numbers ($N < 18$), agreement with experimental results remains acceptable (see fig.10.35). The fact that N has to be a specific integer, however, limits the possibilities of imposing initial conditions.

The fact that the core size of a vorton ring is proportional to the ring's radius, means that stretching of a vorton ring, or more generally a vorton tube, is accompanied by an increase of the core size. Physically, this seems incorrect. This result has been recognized before by others. A vorton division procedure has been proposed to avoid this phenomenon. However, our implementation of vorton division has brought to light some serious drawbacks of this procedure (see the discussion below in §11.1.5).

The discrete representation of continuous vortex configurations, as done in the vorton method, may be criticized. From fig.10.1(b) and fig.10.2 in §10.1 we have found that the distribution of vorticity for a vorton ring in azimuthal direction (i.e. along the torus) is not homogeneous. However, no serious negative consequences seem to be related to this as long as cores do not "touch" [5].

Another important drawback of the discrete representation appears to be the possibility of vortons losing their alignment once vorton tubes approach closely. In fig.10.32 we have seen one example of this behaviour: closely approaching vortons tend to form "dipoles" and leave the main structure. In this regard, vortex-filament methods have to be preferred.

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[3] We have to add that this result may depend on the time steps used. However, our time step adaption scheme appeared perfectly reliable in all cases not involving the K-equation.

[4] Compare, however, a remark by Hou & Lowengrub [90]. They have stated that the "singular Biot-Savart kernel in the [3-D vortex-point method] has a natural cut-off", i.e. if vortex elements have initially been separated a distance h, they will never come closer than a distance proportional to h.

[5] The touching of cores can be defined as the situation in which the distance between two vortons becomes equal to the sum of their core radiuses, defined according to fig.10.2.

11.1.3 Correct Representation of Deformation and Interaction

Vortex Deformation

All simulations performed with application of the N+K-equation have shown correct representation of deformation (rotation and stretching) of vorticity as long as the vorton rings do not approach closer than a certain distance [6]. Especially the simulations presented in §10.5 have shown that for the N- and K-equation correct deformation is not assured.

Core Deformation

We have also observed that the core deforms, which for the elementary case of §10.3 is in accordance with our expectations, at least qualitatively (see fig.10.17 and fig.10.18).

However, from fig.10.17 and fig.10.18 in §10.3 we have found that the core deformation in a vorton ring also depends on the azimuthal position in the vorton ring. For locations between vortons, a rebounding movement of the core has been observed. This behaviour can be regarded as an azimuthal disturbance of the ring, though our numerical simulations did not show any signs of instability [7]. We regard this behaviour as an undesired artefact of the vorton representation. No clear indications exist that the vorton representation really fails at the moment this behaviour sets in.

Stability

Regarding the stability of vorton rings (see §10.1.2), we have found rather good agreement with numerical results by Knio & Ghoniem [108], but poor agreement with experimental results; see fig.10.5. An explanation for this weak performance of the vorton method may be the lack of resolution. For the values of the non-dimensional velocity $\tilde{V}$ for which experimental data are available, the number of vortons in the ring is only little more than twice the unstable wave mode number.

Besides, our representation of the vortex ring by just a single "layer" of vortons may be insufficient to adequately represent the internal core dynamics. Knio & Ghoniem have shown that a multi-layer representation of the vortex ring (similar to that shown in fig.10.28) leads to a better agreement with experimental results. An objection against a multi-layered torus is related to the indefiniteness of the positions of the vortons and their initial strengths. Winckelmans [283] has provided pictures which show the presence of unsteady behaviour within the core, which may be due to leap-frogging of the circular vortex filaments which make up the core of the torus. Though Winckelmans's multi-layered ring seems to be stable, the advantages of his representation (see fig.10.28) above a single-layer representation are not clear.

Our simulations of initially distorted head-on colliding vorton rings (see fig.10.23) suggest that a likely explanation for the small-ring formation as found by Lim (see §10.3.2) can be ascribed to the growth of an unstable wave mode on both rings and an ensuing reconnection process. However, full quantitative comparison with Lim's experiment have proved to be impossible (see note 7). An interesting simulation for this configuration would be one of randomly disturbed rings.

[6] One would be tempted to restate this as: as long as cores do not touch. However, evidence for this statement is lacking, since we have not found a indisputable definition of the core size of a vorton ring; see fig.10.3.

[7] For large numbers of vortons (i.e. large than those used in the simulations presented in Chapter 10), we encountered an apparent instability of the vorton rings. This has been the case in our attempt to simulate Lim's configuration of two head-on colliding vortex rings as described in §10.3.2.

Reconnection
The simulation of §10.4, which has been performed with the aim of investigating the reconnection of two vortex rings as in the experiment by Izutsu & Oshima (IO), may have bewildered the reader as reconnection is generally supposed to be possible only by viscous annihilation of vorticity (see §C of the Interlude).

Nevertheless, the simulation presented in fig.10.32(c)(ii) shows rather good qualitative agreement with the IO experiment. The development in time does not agree exactly, which may be ascribed to differences in the initial configuration. This is shown, for instance, by the difference in development of the angle of inclination θ of the rings.

However, our simulation results do not show convincing evidence for the presence of threads (see e.g. fig.10.29) which have been observed in the IO experiment and also in the simulations by Winckelmans [283] and by Kida *et al.* [103]. Fig.10.30 suggests that the formation of threads is related to the "tails" of vorticity which are formed downstream of the vortex rings. These tails may be due to the initial Gaussian distribution of vorticity in the rings:
for equilibrated initial conditions, Gaussian distributions seem not appropriate and the tails may be due to a reorientation of the cores towards an equilibrium shape. This would imply that they are an artefact of the computational modelling [8].

If a dependency of the reconnection process on the Reynolds number really exists, as Anderson & Greengard have suggested (see §10.4.2), then one might wonder whether the interaction of two vortex rings attains a Re-independent behaviour for large Re and whether our simulation is a representation of this limit case. We suggest that the behaviour shown in fig.10.35 only mimics physical vortex reconnection and is just a consequence of the computational model. A clue to this last statement can be found in the results presented in fig.10.32 and mentioned at the end of §10.4.2. The interaction of two vortex rings at the moment they "touch" depends on the arrangement of the vortons in the rings relative to the point of closest approach [9]. A slight disturbance in the symmetry of the configuration may seriously disturb the "reconnection". Another clue can be found in the simulations discussed in §10.2 (fig.10.8), which have revealed that vorton reconnection does not always occur when experiments suggest it should (e.g. in case of axis ratio $L/(2R_e) \approx 7$).

We see that three important questions arise:

1. Is the reconnection observed in our numerical simulations an adequate representation of the physical process?

2. How can reconnection occur in an inviscid simulation?

3. Why does reconnection occur in case of two obliquely interacting vorton rings and not in case of a pseudo-elliptical vorton ring?

Our answer to the first question has already become clear from the remarks above. However, it can only be answered conscientiously if an extensive comparison is made between the numerical and the experimental results. Unfortunately, the latter are still scarce. Besides, the

[8] See the discussion on models in the Epilogue.

[9] Regarding the position of the vortons, we could wonder whether the "dipole" seen in figs.10.32(i) is a numerical artefact only. The "dipole" does not seem to influence the reconnection of the rings. However, we have seen that it does influence the subsequent behaviour of the reconnected vortex ring, i.e. the presence or absence of splitting into two rings.

vorton method may not be able to allow complete simulation on all relevant scales and reveal the exact mechanisms of this phenomenon [10].

Regarding the second question, we first of all have to remark that no real evidence exists for the impossibility of inviscid reconnection [11]. The rejection of inviscid reconnection by some seems to be based [12] on Helmholtz's First Theorem and its interpretation that vortex lines cannot end inside any volume. As remarked in note 3 of Chapter 2, this result is only true for vortex tubes.

Pedrizetti ([177] and [178]) has suggested that the vorton method implicitly introduces a viscous effect during rapid stretching of the vortons. The "viscosity" in this case is proportional to the rate of stretching and the core size. As the vortons involved in reconnection are strongly stretched, the "viscosity" tends to large values and reconnection can happen. According to Pedrizzetti this is "the mechanics which permits to jump over the moment of local intense stretching as vortex reconnection, which, otherwise, could hardly be followed numerically" [177].

In our opinion, this explanation is dubious. We think that the reconnection of two inclined rings can be explained from the "alignment" behaviour of vortons. To illustrate our interpretation of the apparent reconnection of vorton rings, regard the configuration illustrated in fig.11.1. The two pairs of vortons can be imagined to be each part of a vorton ring as in the configuration of §10.4 (the rings are suggested by means of the dotted lines). When the "rings" approach each other and get deformed, the angle between strength vectors of vortons 1 and 2 (indicated by the arrows) changes and their alignment is weakened. At the same time the angle between vortons 1 and 3 changes and their alignment improves. At a certain moment vorton 1 becomes stably aligned with vorton 3 and the "reconnection" has taken place [13].

The above consideration may also settle the question 3 mentioned above. In the case of the pseudo-elliptical vorton ring, the angle between the vortons 1 and 2 does not reach a critical value at which realignment of vorton 1 with vorton 3 is possible [14].

If annihilation of vorticity due to viscosity is really an important ingredient of the reconnection process, we must seriously doubt the applicability of the vorton method to simulate vortex reconnection. A remedy in this case may be the introduction of a viscous term to the vorton equations as has been proposed by Winckelmans [283] (see §10.4.2).

Several authors (e.g. Lim [129]) have pointed at the appearance and importance of helical vortex lines during vortex reconnection. Since vortex lines have not been visualized in our simulations, we cannot tell whether helical vortex lines have been present. Possibly, this twisting of vortex lines can only be simulated correctly if the vortex ring is represented by the multi-layered torus used by Knio & Ghoniem and by Winckelmans (see above).

[10] In future numerical investigations of vortex reconnection, one should examine the behaviour of other diagnostics: besides the isosurfaces of vorticity, those of the rate of strain and enstrophy production may be used for comparison with experimental data. The visualization of vortex lines may also be informative on the exact mechanism of reconnection. However, it can also be misleading, as Robinson has remarked [196].

[11] One surprising result in this regard is the suggestion by Melander & Hussain in [160] that reconnection occurs on a convective timescale.

[12] See e.g. the quotation from [103] given in §10.4.2.

[13] One could call this a bifurcation, due to the resemblance with this mathematical concept.

[14] As remarked in §10.2.2, another possible explanation may be related to the initial restriction (see fig.10.12), though even in that case no reconnection has been found. Another explanation may be related to the fact that in the vorton representation inertia is not included. The inertia of the approaching parts of the ring after the switch of the axes may be responsible for the close approach which subsequently leads to reconnection.

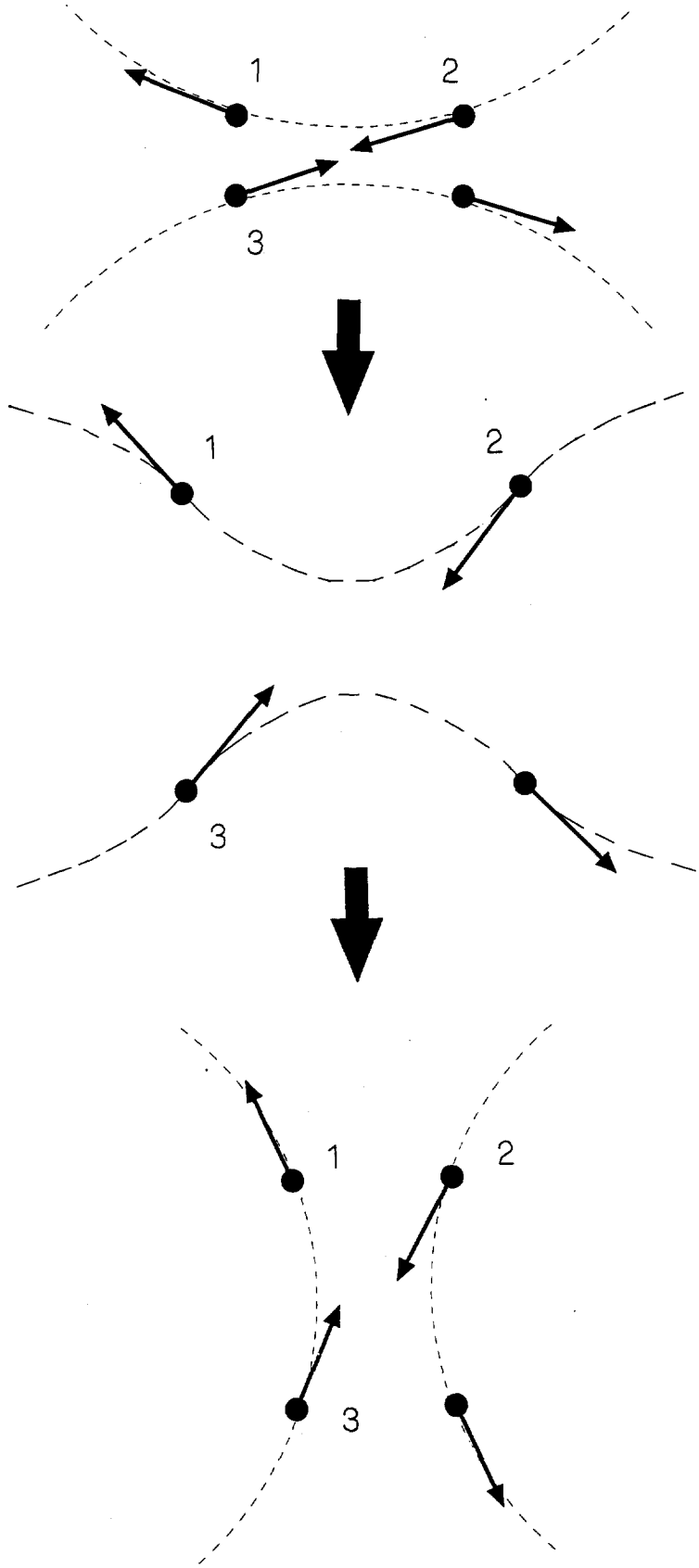

Figure 11.1: Elementary vorton configuration to demonstrate the possibility of "reconnection" (see text). The dotted lines represent parts of vorton rings. Big arrows indicate time development.

Alignment
The simulations in §10.5 of two knotted rings have made clear that the vorton rings correctly represent the tendency of alignment. Shortly after the completion of the alignment, however, the simulation breaks down. Since experimental data on this configuration are lacking, it is impossible to make any further remarks or to draw definite conclusions on the physical correctness of the alignment and on the behaviour following alignment. For instance, we would like to know whether reconnection may be expected in this case. We have implemented several initial positions of the vortons in the rings, but in none of these cases reconnection appeared to be (even dimly) present. Possibly, in real viscous flows the aligned anti-parallel vortex tubes will annihilate each other. In case this is an essential part of this interaction, we have to conclude that the application of the vorton method as presented in this thesis is not warranted.

11.1.4 Conservation of Motion-invariants

In §9.3.1 and Appendix A it is made clear that our expressions used as diagnostics for the simulations can be criticized. This has been a serious obstacle in drawing any conclusions with regard to the question: do vorton simulations show conservation of the relevant motion-invariants?

The simulations in §10.2 (fig.10.9 and fig.10.10) show that the (nonzero) motion-invariants are rather well conserved as long as no severe vortex (core) deformation takes place. However, fig.10.40 in §10.5 suggests that conservation is seriously violated the moment the cores "touch" and severe stretching and deformation takes place. However, fig.10.33 in §10.4 proofs that reconnection does not necessarily mean a strong violation of conservation.

The time-development of interaction-energy E_i and self-energy E_0 have opposite trends (see fig.10.22 and fig.10.33), which suggests that $E_i + \alpha E_0$ (where α is some constant) may be a correct representation of the total kinetic energy.

The simulation in §10.5 (fig.10.40(c)) shows that interaction-helicity H_i is well conserved up to the moment the aligned parts of the vorton rings approach each other closely. The subsequent violation of conservation of H_i may be attributed to the failure of the vorton method to represent annihilation of vorticity, as discussed above.

11.1.5 No Negative Effects of Remeshing (Vorton Division)

As remarked in §11.1.2, vorton division may be essential, since it counteracts the growth of cores of vorton rings during stretching. Besides, addition of vortons increases the resolution in areas where this seems essential for correct simulation. However, as explained in §10.3.3, it may engender undesired effects. Besides, the insertion of vortons leads to an abrupt change in the core size and structure. This may be of serious consequence to the stability of the vortex structure. Therefore, inclusion of vorton division should be implemented very carefully [15]. At least, we can conclude from fig.10.24 in §10.3 that division without updating (see §9.4) has to be rejected. Division with updating improves conservation of interaction-energy.

Some additional remarks have to be made. Division will become troublesome in case of reconnection, due to the possibility of a sudden exchange of neighbours between the vortons. The value of factor λ (see §9.4) has been derived under the assumption of a circular core hence for strong core deformation, this value may have to be adapted. Furthermore, updating

[15] E.g., in the interpolation of the locations of the added vortons (see fig.9.3) use could be made of spline interpolation. For a radially growing vorton ring, as in §10.3, this will assure a conservation of its circularity.

with regard to circulation Γ, as we have done (see §9.4), does not imply conservation of other invariants.

The inability of our vorton method to represent annihilation (see above) also causes the final breakdown of the situation illustrated by fig.10.36, i.e. one of the possible effects of vorton division. If annihilation would take place in this situation, the simulations to which we have applied vorton division might show correct development of the simulations of §10.4 and §10.5, instead of breakdown.

11.1.6 Correct Boundary Conditions

As explained in §9.1, we have chosen for vorton configurations for which it is not necessary to implement explicitly boundary conditions. Only in case of the simulation presented in §10.6 (simulation of a free-slip flat boundary by means of mirrorimaging), this requirement may be important (see §11.2 below for further discussion).

11.1.7 Convergence

In literature, the convergence requirement has been formulated for vortex-point methods by the question [16]: do vortex configurations represented by vortex elements tend to represent continuous vortex structures better and better for an increasing number of vortex elements?

With regard to the vorton method, this question seems irrelevant to us. In §10.1.1 we have shown that increasing the number of vortons implies changing the characteristics of a vorton ring and renders an investigation into convergence, as defined above, impossible [17].

The formulation of the convergence requirement as used in literature and stated above can be replaced by an alternative one. Actually, for a vortex method like the vorton method, we would like to have a proof of a property which has been given for the 2-D point-vortex method. This property is related to the following question: do 2-D vortex points show the same (qualitative) dynamics as patches of vorticity [18] whose behaviour is obtained by directly solving the Euler (Helmholtz) equation? This has been shown both analytically by Marchioro & Pulvirenti [141] and numerically by Benzi *et al.* [23]. For the vorton method, such a proof does not seem feasible in the same way since 3-D patches cannot exist on their own. Instead,

[16] Note that we give only an informally expressed version of this question here. An exact mathematical formulation can be found in literature, e.g. [90].

[17] Despite this situation with regard to the vorton method, several authors have suggested that their proofs of convergence apply to this point-vortex method. However, partly because of their mathematical nature, it is hard to find out whether the results indeed apply to the (soft-)vorton method.

For the soft-vorton method (see Appendix B) convergence seems to have been investigated first by Cottet [41], who proved that the appropriate error norm for the velocity and vorticity fields goes to zero as the number of soft vortons increases and the core-size decreases subjected to the constraint that the cores overlap (i.e. the core sizes have to be larger than the typical distances between the elements). Another proof of convergence for this case has been given by Beale [21]. Winckelmans [283] has shown convergence for the soft-vorton method by means of his numerical simulations, though it appeared to be slow.

Cottet [42] has also shown that his vortex method discussed in [41] converges even without smoothing, thereby apparently providing a proof of convergence for the vorton method. However, the proof required two mathematical tools whose applicability with regard to the vorton method are unclear. Hou in [10] has remarked that the result found by Cottet does not mean that the vortex-point method can be applied without smoothing or desingularization. According to Hou, for any given time T a condition exists for which the method is stable and convergent. However, the number of particles is finite, so there will be a time beyond which particles are so close that stability analysis breaks down. Beyond this time, some "regularization" (i.e. remeshing) is needed.

We have to conclude that a proof of convergence for the vorton method still seems to be absent. However, carefully performed numerical simulations may give valuable clues with regard to this issue.

[18] By patches we mean compact distributions of vorticity.

a close comparison between the dynamics of a vorton ring and a full numerical simulation of a vortex ring will be necessary to serve the same purpose.

11.1.8 Computational Effort

In general, we can remark that computational times for our simulations have been satisfactory (in the order of minutes). However, we must add that these simulations have only been done for relatively small numbers of vortons. The relation between computational times and the number of vortons N has been investigated for the simulations discussed in §10.4. We have found that times are proportional to about N^2.

Simulations of multi-layered rings like those performed by Winckelmans on the configuration shown in fig.10.28 require a much larger effort [19]. Kascic [101] has suggested the use of a vector processor to simulate the dynamics of large numbers of vortons.

11.2 The Vorton Method and research on Coherent Structures

The results of the simulation on the single vorton ring in a shear flow above a flat plate, as presented in §10.6, may be too elementary to allow any conclusions with regard to the applicability of the vorton method to the study of CS in turbulent boundary layers (TBL).

First of all, as indicated in §10.6 experimental evidence for the existence and role of vortex rings or related vortex structures in the TBL is scarce. Furthermore, we do not know whether phenomena like vortex reconnection and annihilation of vorticity are (crucially) involved in the behaviour of any such structures. If this would turn out to be the case, we have to realize that our vorton simulations have shown the inadequacy of the vorton method on this point.

One may also object that the no-slip boundary condition at the surface of the wall and the related generation of secondary vorticity may be crucial for the flow phenomena observed in the wall region of the TBL as some authors have suggested (see §10.6.1).

All the same, our results may illustrate that even elementary and crude configurations can contribute to an understanding of TBL flows [20]. At least, our simulation have shown that an outer region parameter (*in casu* the outer layer velocity U) of the shear flow determines the behaviour of the vorton ring and (consequently) the Reynolds stress pattern in the flow. This suggests that outer layer parameters in the TBL (partly) determine its characteristics.

11.3 Final Remarks

We think that the simulations presented here have given some indication of the applicability of our vorton method (i.e. applying the N+K-equation). We conclude that the vorton method produces simulation results which agree fairly well with experimental and analytical results, at least when vortex structures, like vortex rings, do not approach each other closer than a certain distance. When vorton structures approach more closely (and e.g. viscous effects are likely to become involved), we have to be very careful in judging the numerical results. Especially the simulations in §10.3 and §10.5 have shown that vorton behaviour may start to become chaotical . However, lack of experimental results prevents more decisive conclusions.

The vorton method may be extended to improve its performance. The use of soft-vortons, of multi-layered vorton rings and the addition of viscous diffusion to the vorton equations are possibilities. However, the first option shows important disadvantages and has nowhere been shown to perform better than the ordinary vorton method (see also Appendix B). For

[19] Winckelmans did not provide details on his computational times; the numbers of vortons he typically used were of the order $10^3 - 10^4$.

[20] For a discussion of the nature and use of modelling in turbulence, we refer to the Epilogue.

the second extension, the correct implementation of such rings is still unclear and simulations require a large computational effort. As for the third option, we only have the results by Winckelmans [283]; the same kind of objections exist as for the second option. Besides, a more careful incorporation of vorton division than that applied in our simulations is necessary to increase the applicability of the vorton method.

The vorton method is a relatively cheap, quick, and simple to handle vortex method, able to provide a first indication of the behaviour of vortex configurations [21]. However, for a really careful simulation of closely interacting vortex structures, the method may not be reliable and the application of a viscous vorton method (like that of Winckelmans) may be more appropriate. Besides, still other numerical (vortex) methods unrelated to the vorton method may be better suited for certain simulations [22].

[21] We agree with Chorin's remark: "a good guess at the solution of the problem one wants to solve is better than an unambiguous solution of the wrong problem" [37].

[22] One recent promising method is that by Verzicco and co-workers, who solve the Navier-Stokes equation by means of a finite-difference scheme (see e.g. [273]).

Epilogue

In this final chapter, I will attempt to bring together the vortex-atom-part and the vorton-part on a scientific-philosophical level. In the preceding chapters I have shown how both parts are related on the scientific level, e.g. by showing how the theorems and equations first proposed by Helmholtz and Kelvin can be applied for the derivation of the vorton equations, Besides, in the Interlude I have indicated (though only in a superficial manner) how certain aspects of vorticity theory show a continuous development from the days of the vortex atom to the present.

On the scientific-philosophical level, the vortex atom and the vorton are not related in such a direct sense, though at least I will quote some of the 19th century authors who have been mentioned in the vortex-atom-part [23]. The relation I would like to discuss is based on a common and important concept involved in both the vortex-atom theory and the vorton theory: the **model**.

Everyone familiar with any part of science will have some notion when reading this term. Even restricting the discussion to models in physics, it appears difficult to formulate an unambiguous description of this term. I define a model as a representation of a physical concept [24] that is still unknown in details, but of which one has some image. The model is a simplification of reality, but tries to catch the essential aspects of the real concept. This description rouses questions with regard to the meaning of "reality". Here, I will equate reality to experimental observations.

Several kinds of models may be discerned [25]:

- **analytical models**

  These models consist of (sets of) equations which are supposed to describe in mathematical terms the physical concept which has to be modelled. They do not necessarily contain any viewpoints on the physical backgrounds of the concept.

  Examples are Maxwell's famous equations describing electrodynamical phenomena (see §6.3) and Saffman's model of reconnection (shortly mentioned in §C of the Interlude).

- **physical models**

  A physical model is supposed to be a direct representation of some aspects of the physics involved in the concept to be modelled.

  Example of physical models are the vortex atom model as proposed by Kelvin and the model of a coherent structure (CS) in a turbulent boundary layer as presented in §10.6. Both examples will be fully discussed in this Epilogue.

- **conceptual models**

  In the case of conceptual models one does not suppose that the ingredients used in the model necessarily form a real physical representation of the concept to be modelled (in

[23] More often than is usual nowadays, these scientists occasionally discussed the philosophical backgrounds of their own and others' research.

[24] By physical concept I mean anything which physicists tend to model: objects, phenomena, processes, etc.

[25] Here, as everywhere else in this Epilogue, I will use my own terminology. This list is not exhaustive.

contrast with physical models). The conceptual model and reality have to show similar properties, but the model does not necessarily represent the physical background of the concept.

As an example of this type of models I mention the mechanical (or mechanistic) models favoured by British scientists (e.g. Kelvin) in the second half of the 19th century (see the introduction of Chapter 3 and of Chapter 5). They essentially amounted to the representation of physical concepts by means of a "mechanism" involving springs, wheels, gyroscopes, etc. The laws of (classical) mechanics determined then the behaviour of these models [26].

The aim of a model is to aid in the visualization (and possibly quantification) of physical phenomena and in the understanding of the physical "mechanisms" which determine the character of a physical concept. Some remarks on models can be found in one of the first essays on the use of models, i.e. Rankine's discussion [185] of his own conceptual model of matter, the molecular vortices (see §3.1). According to Rankine a model (or "hypothesis" as he called it)

> substitutes a supposed for a real phenomenon, ... the object being to deduce the laws of the real phenomenon from those of the supposed one. If the supposed phenomenon were more complex, or less completely known in its laws than the real one, the hypothesis would be an incumbrance, and worse than useless. ...
>
> A hypothesis is absolutely disproved by any facts that are inconsistent with it. ... On the other hand, no hypothesis is capable of absolute proof by any amount of agreement between its results and those of observation; such agreement can give at best only a high degree of probability to the hypothesis. ...
>
> The agreement should be mathematically exact, to that degree of precision which the uncertainty of experimental data renders possible, and should be tested in particular cases by numerical calculation. The highest degree of probability is attained when a hypothesis leads to the prediction of laws, phenomena, and numerical results which are afterwards verified by experiment. [185, p.127]

Though Rankine regarded his own hypothesis of molecular vortices as respecting these rules, he warned that hypotheses like these "never can attain the certainty of observed facts" [185, p.132].

On the final fate of models we can read in Larmor's address to the section of Mathematical and Physical Science at the 1900 meeting of the British Association: "When a physical model of concealed dynamical processes has served this kind of purpose ..., when its content has been explored and estimated, and has become familiar through the introduction of new terms and ideas, then the ladder by which we have ascended may be kicked away, and the scheme of relations which the model embodied can stand forth in severely abstract form" [117, p.626].

Naturally, a single phenomenon may be represented by several models or kinds of models. This has been the situation in British science in the latter half of the 19th century. As Duhem remarked in [51] (see §5.2), the British proposed one model for one group of laws and another

[26] Another example may be the model of turbulence proposed by Synge & Lin (see §B of the Interlude) in which the interaction of vortices is supposed to provide characteristics similar to those of turbulent flows. However, it is not clear whether they meant this as a conceptual or as a physical model.

completely different model for another group, though both groups contained some common laws [27].

For Kelvin, around the time of his 1884 Baltimore lecures, the use of models (usually of conceptual nature) was of fundamental importance: "It seems to me that the test of 'Do we or not understand a particular subject in physics?' is, 'Can we make a mechanical model of it?'" [99, p.111] and: "I never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand" [99, p.206]. However, he also stated that his models by no means reflected reality. They were only imitations of reality and certainly not unique.

To further explore Kelvin's use of models and to illustrate the problems related to models, let me now concentrate on his vortex atom model. It is important to notice that the vortex atom theory actually involves two kinds of modelling.

Whereas for his mechanical models Kelvin did not claim a reflection of reality, with regard to the relation between vortex atoms and matter, Kelvin suggested that the *physical* vortex ring, as seen in Tait's experiment, was a physical model of the atom. To confirm this opinion, I refer to the account of Kelvin's 1867 lecture "On Vortex Atoms" [243] (see §4.2), where we read:

> After noticing Helmholtz's admirable discovery of the law of vortex motion in a perfect liquid ... the author [=Kelvin] said that this discovery inevitably suggests the idea that Helmholtz's rings are the only true atoms. [243, p.1]

This suggests that initially Kelvin indeed regarded the vortex atom as a physical representation of the atom. However, I think that many of his contemporaries could only regard it as a conceptual model, as is expressed by Larmor in the quotation given at the end of §6.3.

In order to demonstrate the correctness or usefulness of his (physical) model, Kelvin had to show that it possessed the properties of "real" atoms. That is to say, the properties which were known at that time. As he must have realized that this would be difficult to show experimentally and unconvincing, he went for the analytical elaboration of another type of model of the vortex ring itself. However, according to the definition given above this model cannot be called an analytical model and I shall name it a **computational model**. For Kelvin, initially *the* computational model of the vortex ring was the Kelvin-ring (see §A.2 of the Interlude).

Regarding the story of the vortex atom, I conclude that for Kelvin and his contemporaries, the vortex atom failed as a physical model, since it appeared to lack some of the fundamental properties related to real atoms (stability, gravity, spectra; see §5.3). However, regarding the development of the vortex atom from the present point of view, one can claim that the vortex atom suffered from another weakness. Today, we know that the Kelvin-ring is only a crude computational model of a vortex ring (see §A.2 of the Interlude). At that time, however, this

[27] Duhem stressed that it is better to have one unique theory since this provided the "classification naturelles des lois" and showed the order in nature. On the other hand, he argued that such a situation as in British physics had to be allowed: "Si l'on astreint a n'invoquer que des raisons de logique pure, on ne peut empêcher un physicien de représenter par plusieurs théories inconciliables soit des ensembles divers de lois, soit même un groupe unique de lois; on ne peut condamner l'incohérence dans le développement de la theorie physique" [51, p.366].

circumstance did *not* contribute to the fall of the vortex atom since this computational model was generally regarded sufficient. If, for example, Kelvin would have applied the analytical techniques of Widnall and co-workers (see §A.3 of the Interlude), he could have demonstrated the inherent instability of the Kelvin-ring and would have been forced to revise his computational model. Kelvin's eventual recognition of this weakness can be deduced from his 1889 remark on Hicks's hollow vortex (see §6.3). However, by then the vortex atom had already appeared unviable as a *physical* model.

The story of the vortex atom shows us other factors which may influence the viability of any model [28]:

- In general, one can say that in the case of the vortex atom there has been no rational line of defense. The elaboration of the model (both with regard to its role as physical and as computational model) lacked a real research program. Besides, no attempts were made to refute fundamental criticism, such as that by Reynolds (see §5.1).

- The promoters of the vortex atom theory did not provide quantitative data, which meant that comparison with experimental data was impossible. If more fundamental experiments would have been done on the properties of vortex rings (e.g. on their velocity, distribution of vorticity in the core), the computational part of the model would have been discredited. If quantitative comparison with properties of matter (e.g. spectral lines; see Julius's contribution discussed in §5.3.4) would have been performed seriously, the physical part of the model would have been discredited.

- Relatedly, the model was not predictive (see Rankine's remarks quoted above). Therefore, the model must have appeared useless and irrelevant and could not show any advantage above other models.

- The nature of the model was purely hydrodynamical, while experiments eventually showed that other aspects (e.g. electrical) were essential for an atom model. Therefore, it is not surprising that the vortex atom did not survive the discovery of the electron and of radioactivity [29]. Attempts to adapt the vortex atom model (e.g. by introducing hollow cores; see §5.1), or to extend it (e.g. with electric charge; see §6.2) could not prevent its fall.

- The computational model was hard to elaborate due to lack of sufficient mathematical techniques. Though this was fully recognized by the promoters of the theory and though they introduced several new techniques, progress was slow and several important issues could not be tackled properly.

However, the vortex atom model did not only fail as a result of internal inconsistencies. It also suffered from the shift in the use of models which occurred at the end of the 19th century in British physics (see §6.3). When the vortex atom had started to decline, Kelvin complained that Maxwell's use of analytical models had superseded his own use of physical and conceptual models. Maxwell's initial emphasis on analogy and heuristic models (e.g. his molecular vortices; see §5.1) had changed towards an approach according to which physical phenomena were framed into mathematical equations, i.e. the use of analytical models. To

[28] The order of these factors does not indicate their relative importance.

[29] Likewise the vortex ether did not survive relativity.

Kelvin Maxwell's equations were examplary for the wrong approach. They were "metaphysical" and had been worked out in the mind without contemplation of physical reality. Today, we can conclude that Maxwell's analytical model has been much more successful than Kelvin's physical and conceptual models.

Taking into account the factors mentioned above, it may seem remarkable that the vortex atom model *could* survive for almost 30 years. However, this can be attributed to several favourable circumstances. First, competing theories of matter lacked the same or other fundamental and practical problems. Secondly, lack of experimental data on the properties of matter prevented a definite judgement. Thirdly, the crucial role of electric charge in matter became only fully realized after J.J. Thomson's 1897 discovery of the electron. And last, but not least, Kelvin's fame must have played some role here.

In conclusion, one can say that many factors are involved in the development of a model. Some are evident, others are not. Some can be analyzed rationally, for others this seems impossible. I think that all factors mentioned above, both favourable and unfavourable to the viability of models, can still be found today. Some evidence for this statement can be found in the next and last part of this Epilogue.

In this last part, I will treat two analogies I have found between the vortex-atom-part and the vorton-part with regard to the use of models:

- computational modelling:

  Kelvin ring $\leftrightarrow$ vortex ring $\quad\sim\quad$ vorton ring $\leftrightarrow$ vortex ring

- physical modelling:

  vortex atom theory $\leftrightarrow$ matter $\quad\sim\quad$ vortex models of a CS $\leftrightarrow$ turbulence

As mentioned above, one of the obstructing factors in the development of the vortex atom model has been the lack of proper mathematical techniques. In modern fluid dynamics research the use of numerical techniques has proven to be an important and fruitful new tool to elaborate models. Vortex methods (see Chapter 7) form one part in this field of so-called computational fluid dynamics. Despite the progress in computational capabilities provided by these methods, one still needs a computational model of vortex structures, e.g. vortex rings, on which to apply the numerical tools. In our investigation of the vorton method, the computational model of the vortex ring has been the vorton ring as illustrated in fig.9.1. In Chapter 10 I have investigated the correctness of this model by means of numerical simulations. From the discussion in Chapter 11 the reader may have deduced that the usefulness of the vorton ring as a computational model can be questioned [30]. Moreover, as has been the case for the vortex atom, a good comparison of numerical with experimental results may be impossible, not only due to the scarcity of experiments but also due to the fact that viscosity may have an essential influence on experimental rings.

The second analogy, related to physical modelling, can be found in the vorton simulation treated in §10.6. There, I discussed the present trend in fluid mechanics to regard the role of coherent structures in turbulent flows, *in casu* turbulent boundary layer flows. As discussed in §10.6.1, some have suggested that these structures can be modelled as vortex rings. Notice that this physical modelling is different from that in case of the vortex atom. The vortex

[30] For large numbers of vortons in the vorton ring, it even approaches the Kelvin-ring and we may expect the same problems of modelling as in case of the vortex atom.

ring, Kelvin supposed, could be completely *identified* with the atom. Today, the vortex ring is regarded as an essential part of turbulent flows and is not identified with anything else; it is just a vortex ring.

Turbulence modelling has a history showing important shifts in approach (see also §B of the Interlude). After Kelvin's and FitzGerald's 1887 vortex ring model, at the beginning of this century modelling of turbulence had become largely analytical. In the 1930s the statistical approach began to dominate research in turbulence, which lacked the use of models. Only in the 1950s one realized again the importance of modelling and the concept of "coherent structures" was introduced. Nowadays, research on CS shows a large variety of (vortical) structures which are proposed as explanation for physical phenomena in turbulent flows.

Besides these typically physical models, it should be mentioned that today several other types of modelling are used in turbulence research. This can only be encouraged, as Duhem already realized (see above). Lumley in [135] commented on the ability of turbulence models (including statistical methods) to increase our understanding of turbulence: "However, I believe it is foolhardy to expect them to. These models are simply embodyment of experience; they are something constructed to behave like turbulence, in situations where it has been observed, to be used as a design tool. A model cannot, except by accident, contain more than is put into it."

Naturally, this last remark is relevant to any kind of model. Because electric charge had not been put into the vortex atom model, it was unable to model the atom. In our present models of turbulent flows, we should strive for models which can surpass, so to speak, limits. The main problem will be how to set up such models and how to interpretate their results. Some models may suggest "too much" but this is not important; the point lies in suggestion, not demonstration [31]. Nevertheless, one has to be alert that models may become more important than the phenomena themselves.

We can only hope that the models of CS will improve our understanding of turbulence as a physical phenomenon. The central question in this respect has been formulated by Kline & Robinson in [73] as: "how do we capture the essence of such a model in a simple enough way so that it becomes useful in creating predictive models?" We must realize that even negative results can help us and that the road towards complete understanding, if ever achieved, certainly isn't straight.

As in the case of the vortex atom, the most important problem here is the relatively small amount of knowledge on the characteristics of CS and their role in the TBL from experiments. We even have a lack of definition and problem formulation (as already remarked in §B of the Interlude). As long as this situation lasts, models cannot be judged correctly.

Though of enormous help in advancing our knowledge on fluid flows, the present trend of computational fluid mechanics brings along its own problems. For the specific case of vortex methods, the computational modelling of vortices (as discussed above for the vorton method) should be performed very carefully. In addition to this, other problems arise due to the large amount of data provided by numerical simulations. The problem of selecting those data which are useful for the purpose of understanding will only become more urgent. And, relatedly, we have to face the problem also formulated by Kline & Robinson in [73]: "how do we combine the results from numerical simulation data bases with experimental results to approach consensus on a complete model of structure [in turbulence]?" Another problem related to computational methods are the possible influences of numerical artefacts (e.g. numerical viscosity). These

[31] Compare the suggestion of the vortex atom to J.J. Thomson in his discovery of the electron (see §6.3).

may disturb our view on the real physical value of models.

One may think differently about the fate of models, but surely the energy put into their elaboration will not be lost, even if the picture they provide does not correspond to reality or only slightly. Models can give impulses towards new developments and the mathematical topics which they induce may well be worth treatment themselves. Though the vortex atom model itself failed, it left behind a heritage: it meant an important stimulus to the research on vortex motion and led to Tait's contribution and foundation of the theory of knots (see §C of the Interlude).

To conclude, I remark that a model must be used as a first step in investigating the physics of a phenomenon. Afterwards, experimental and analytical results have to be invoked to lay down a theory. At that time, Larmor's "ladder" may be kicked away. Kelvin already realized the relative value of models and eventually left the vortex atom for new models in which he inserted new concepts arising in physics. Perhaps, one day, we have to recognize that our present approach to the modelling of turbulence is unfruitful or should be improved. Then, we must dare to shift towards new approaches.

Appendix A

Vector Potentials and Motion-Invariants

From an elaboration of an *divergence-free* vector potential field, conclusions can be drawn regarding its highest order term and its relation with the conservation of motion-invariants [1].

It is assumed that the vorticity field $\boldsymbol{w}(\boldsymbol{x})$ decays fast enough:

$$\frac{wL}{U} \sim [\frac{L}{x}]^N \text{ as } \frac{x}{L} \to \infty \tag{A.1}$$

where $w \equiv \mid \boldsymbol{w} \mid$, L is a typical length scale, and U is a typical velocity scale. This condition is certainly fulfilled if vorticity decays exponentially [2].

For a divergence-free $\boldsymbol{A}$ we have the Poisson equation:

$$\nabla^2 \boldsymbol{A} = -\boldsymbol{w} .$$

For the far field condition for the velocity field $\boldsymbol{v}$

$$v \to 0 \text{ as } x \to \infty$$

we have the solution

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi} \int_{V'} \frac{\boldsymbol{w}(\boldsymbol{x}')}{\mid \boldsymbol{x} - \boldsymbol{x}' \mid}$$

where V' is the vorticity-containing volume. By expanding this integral in powers of r^{-1} ($r \equiv \mid \boldsymbol{x} \mid$), we get:

$$\boldsymbol{A} = \sum_{n=0}^{m} \boldsymbol{A}^{(n)} + O(r^{-m-2})$$

where

$$\boldsymbol{A}^{(n)} = \frac{1}{4\pi r^{n+1}} \int_{V'} \boldsymbol{w}(\boldsymbol{x}')[(r')^n P_n(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}}')]$$

where $\hat{\boldsymbol{x}} \equiv \boldsymbol{x}/r$ and P_n are Legendre functions.

The first three terms of this sum are given by:

$$\begin{aligned}
\boldsymbol{A}^{(0)} &= \frac{1}{4\pi}\frac{1}{r} \int_{V'} \boldsymbol{w} \\
\boldsymbol{A}^{(1)} &= \frac{1}{4\pi}\frac{\partial}{\partial x_i}(\frac{-1}{r}) \int_{V'} x_i \boldsymbol{w} \\
\boldsymbol{A}^{(2)} &= \frac{1}{8\pi}\frac{\partial}{\partial x_i x_j}(\frac{1}{r}) \int_{V'} x_i x_j \boldsymbol{w} .
\end{aligned}$$

[1] This disucssion is based on [264]; see also [205, §3.2].

[2] Compare this condition with that given in (9.4).

The (far field) description of the vector potential $\boldsymbol{A}$ to the order r^{-m-1} is defined by the nth moments of vorticity for $n \leq m$. These moments of vorticity are defined by:

$$\int_V \boldsymbol{w} \prod_{i=1}^{3} x_i^{j_i} \quad \text{with } j_i \geq 0 \text{ and } \sum_{i=1}^{3} j_i = n \,. \tag{A.2}$$

All nth moments exist for any $n \leq N-3$, where N is the order of the far vorticity field defined in (A.1).

Using this condition (A.1), it can be shown that an nth coaxial moment of a divergence-free vorticity field $\boldsymbol{w}$ along an axis parallel to any vector $\boldsymbol{b}$ should vanish [264, §1.2], i.e.:

$$I^{(n)}(t, \boldsymbol{b}) \equiv \int_V (\boldsymbol{x} \cdot \boldsymbol{b})^n \boldsymbol{w} \cdot \boldsymbol{b} = 0$$

for $t \geq 0, n = 0, 1, 2, \ldots$.

From this result, we can derive consistency conditions, which are linear combinations of the nth moments of vorticity:

- For $n = 0$, we have the consistency condition:

$$\boldsymbol{\Omega} \equiv \int_V \boldsymbol{w} = 0 \,.$$

  This result expresses conservation of the total vorticity. It also shows that the highest order term of the vector potential $\boldsymbol{A}$ disappears, i.e.

$$\boldsymbol{A}^{(0)} = 0$$

  which means that the highest order term of $\boldsymbol{A}$ is r^{-2}. Apparently, total vorticity is always zero for a bounded flow on whose surface $\boldsymbol{w} \cdot \boldsymbol{n} = 0$.

- For $n = 1$, we have the consistency condition:

$$\int_V x_i w_j + \int_V x_j w_i = 0 \;\; (i, j = 1, 2, 3 : j \leq i) \,.$$

  From the required far-field behaviour of the fields $\boldsymbol{v}$ and $\boldsymbol{w}$, it follows that the first moment of vorticity (see (A.2)) is time invariant, i.e.

$$\frac{\partial}{\partial t} \int_V x_i \boldsymbol{w} = 0 \,.$$

  Using the above consistency condition, we then derive:

$$\boldsymbol{P} \equiv \int_V \boldsymbol{x} \times \boldsymbol{w} = \boldsymbol{P}_0 \tag{A.3}$$

  where $\boldsymbol{P}_0$ is the constant vector specified by the initial data. This result expresses conservation of the total linear momentum.

- For $n = 2$, we can derive from the consistency conditions:

$$\boldsymbol{J} \equiv \int_V x^2 \boldsymbol{w} = \boldsymbol{J}_0 \tag{A.4}$$

where $\boldsymbol{J}_0$ is the constant vector specified by the initial data. This result expresses conservation of the total angular momentum.

Combining this relation with the consistency conditions related to $\boldsymbol{P}$, we find another expression for motion-invariant $\boldsymbol{J}$:

$$\boldsymbol{J} = \frac{1}{3} \int_V \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{w}) \, . \tag{A.5}$$

- No additional invariants for $n \geq 3$ have been found [264, §1.2].

Now we regard the vorton fields, presented in §8.2. The vector potential field (8.8) chosen in §8.2 is of order r^{-1} and thus violates the condition derived above. However, since this field is not divergence-free, the above discussion isn't necessarily applicable here. On the other hand, for our vorton vector potential (8.8):

$$\nabla \cdot \boldsymbol{A} = \frac{-1}{4\pi} \sum_\alpha \frac{\boldsymbol{R}_\alpha \cdot \boldsymbol{\gamma}_\alpha}{R_\alpha^3} \, .$$

Hence, $\boldsymbol{A}$ may be divergence-free at an infinite number of points. At these points, the above discussion *is* relevant and defence of the choice of our vorton vector potential (8.8) seems untenable.

The condition on the order of a divergence-free vector potential *is* fulfilled for the Chefranov vortex-dipoles (see e.g. [36]; see for a discussion [206]). The vortex dipoles can be regarded as (infinitesimal) vortex rings. Chefranov has claimed that the equations for their dynamics have a Hamiltonian structure and that his method satisfies conservation of linear motion-invariants. However, no numerical simulations seem to have been performed applying this dipole method.

However, these dipoles may not be suitable for numerical simulations as done in Chapter 10, due to their self-velocity [3]. Synge & Lin [225] (see §B of the Interlude) have investigated the interaction of dipoles in their search for a model of turbulence, but concluded it had "undesirable features" and did not lead to correct correlation functions.

[3] Possibly this self-velocity can be eliminated by the addition of swirl, as has been proposed by Moffatt in [162] in the context of his alternative "vorton" model (see e.g. [158]). According to Moffatt, the original vorton is not a useful concept, as it is no solution of the Euler equation (private communication). The vorton should be a "structure of compact support" that propagates with self-induced velocity and without change of structure and can be regarded as a generalization of the vortex ring. Turbulence, he suggested in [135], could perhaps be regarded as a "sea of interacting vortons". Unfortunately, this theory has not been elaborated yet.

Appendix B

The Soft-Vorton Method

One of the aspects of the vorton method which has been criticized is the singular behaviour of its velocity and vorticity fields. Therefore, Kuwabara [114] has proposed to replace the delta-functions in the original vorton vorticity field (8.5) by smooth functions [1]:

$$\boldsymbol{w}_\sigma(\boldsymbol{x},t) = \sum_\alpha \boldsymbol{\gamma}_\alpha(t)\, \zeta_\sigma(\boldsymbol{x} - \boldsymbol{r}_\alpha(t)) \tag{B.1}$$

where a choice has to be made for the so-called smoothing function ζ_σ for which we require:

$$\zeta_\sigma(\boldsymbol{x}) \to \delta(\boldsymbol{x}) \;\text{ as }\; \sigma \to 0\,. \tag{B.2}$$

This function contains a parameter σ, which can be regarded as a "core radius" of the smoothed vortons.

As remarked in §7.3.3, Kuwabara's soft-vorton method is an example of the Smoothed Vortex-Point Methods. A more general treatment of its theory has been provided by Winckelmans [283], who applied this vortex method in several numerical simulations. The simulations by Winckelmans [283] for the configuration of §10.4, have showed similar results for the original vorton method and the soft-vorton method, both with regard to conservation of diagnostics and to reconnection behaviour.

We shortly repeat two drawbacks of the soft-vorton method, which have already been mentioned in the introduction of Chapter 8. First, the field (B.1) is not divergence-free, like the original vorton vorticity field (8.5). Secondly, several smoothing functions ζ_σ can be applied under the imposed requirements, presumably leading to different simulation results.

The first drawback can easily be suppressed as in the case of the vorton method by deriving a divergence-free vorticity field from an appropriate vector potential $\boldsymbol{A}_\sigma$ [2].

We start from (compare with (8.8)):

$$\boldsymbol{A}_\sigma(\boldsymbol{x},t) = \sum_\alpha \phi_\sigma(\boldsymbol{R}_\alpha,t)\, \boldsymbol{\gamma}_\alpha(t)\,.$$

This time, the function ϕ_σ remains undetermined and is a function of σ. In the manner shown in §8.2, we derive a velocity field:

$$\begin{aligned} \boldsymbol{v}_\sigma(\boldsymbol{x},t) &= \sum_\alpha \boldsymbol{\nabla}\phi_\sigma \times \boldsymbol{\gamma}_\alpha \\ &= \sum_\alpha (\boldsymbol{\nabla}\phi(\boldsymbol{R}_\alpha) \times \boldsymbol{\gamma}_\alpha) g_\sigma(\boldsymbol{R}_\alpha) \\ &= \frac{1}{4\pi} \sum_\alpha \frac{\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_\alpha}{R_\alpha^3} g_\sigma(\boldsymbol{R}_\alpha)\,. \end{aligned} \tag{B.3}$$

[1] A similar vortex method has been proposed by Mosher [165].

[2] The σ will indicate a soft-vorton function or field.

where ϕ is the function defined by (8.9). And from this velocity field we derive the vorticity field:

$$\begin{aligned} \boldsymbol{w}_\sigma(\boldsymbol{x},t) &= \sum_\alpha \{\boldsymbol{\gamma}_\alpha \zeta_\sigma(\boldsymbol{R}_\alpha)) + \boldsymbol{\nabla}(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{\nabla}\phi_\sigma)\} && \text{(B.4)} \\ &= \frac{1}{4\pi}\sum_\alpha \{\boldsymbol{\gamma}_\alpha \zeta_\sigma(\boldsymbol{R}_\alpha) - \boldsymbol{\nabla}[\frac{\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_\alpha}{R_\alpha^3} g_\sigma(\boldsymbol{R}_\alpha)]\} && \text{(B.5)} \\ &= \frac{1}{4\pi}\sum_\alpha \{[\boldsymbol{\gamma}_\alpha - \frac{(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_\alpha)\boldsymbol{R}_\alpha}{R_\alpha^2}]\zeta_\sigma(\boldsymbol{R}_\alpha) - \boldsymbol{\nabla}[\frac{\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_\alpha}{R_\alpha^3}] g_\sigma(\boldsymbol{R}_\alpha)\} \,. \end{aligned}$$

In these derivations, use has been made of the following relations between the functions $\bar{\phi}_\sigma(\rho) \equiv \sigma\phi_\sigma(\boldsymbol{x})$, $\bar{g}_\sigma(\rho) \equiv g_\sigma(\boldsymbol{x})$, and $\bar{\zeta}_\sigma(\rho) \equiv \sigma^3\zeta_\sigma(\boldsymbol{x})$ (where $\rho \equiv \boldsymbol{x}/\sigma$):

$$\begin{aligned} -\nabla^2\bar{\phi}_\sigma(\rho) &= \bar{\zeta}_\sigma(\rho) \\ \bar{g}_\sigma(\rho) &= -\rho^2\bar{\phi'}_\sigma(\rho) \,. \end{aligned}$$

If we impose the condition that the function $\bar{\zeta}_\sigma$ satisfies the following normalization (convergence) condition:

$$4\pi\int_0^\infty \bar{\zeta}_\sigma(\rho)\rho^2 ds = 1 \,,$$

we find:

$$g(\rho) \to 1 \text{ as } \rho \to \infty \,. \qquad \text{(B.6)}$$

Consequently the field (B.3) converges towards the vorton velocity field (8.10) for $\sigma \to 0$.

From the soft-vorton fields derived above, the soft-vorton displacement and deformation equations can be derived.

The displacement equation is easily obtained from the velocity field (B.3):

$$\begin{aligned} \dot{\boldsymbol{r}}_\alpha &= \boldsymbol{v}_\sigma(\boldsymbol{r}_\alpha,t) && \text{(B.7)} \\ &= -\sum_\beta (\boldsymbol{\nabla}\phi_\sigma(\boldsymbol{R}_{\alpha\beta}) \times \boldsymbol{\gamma}_\alpha) g_\sigma(R_{\alpha\beta}) && \text{(B.8)} \\ &= \frac{1}{4\pi}\sum_\beta \frac{\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_{\alpha\beta}}{R_{\alpha\beta}^3} g_\sigma(\boldsymbol{R}_{\alpha\beta}) \,. && \text{(B.9)} \end{aligned}$$

For the derivation of the deformation equation of a soft vorton α we can make use of the splitting of fields mentioned in §8.3 (see also Appendix C): $\boldsymbol{v}_\sigma = \boldsymbol{v}_\sigma^\alpha + \tilde{\boldsymbol{v}}_\sigma^\alpha$ and $\boldsymbol{w}_\sigma = \boldsymbol{w}_\sigma^\alpha + \tilde{\boldsymbol{w}}_\sigma^\alpha$. We then have to elaborate (in case of the ordinary representation of the Helmholtz equation):

$$\frac{D(\boldsymbol{w}_\sigma^\alpha + \tilde{\boldsymbol{w}}_\sigma^\alpha)}{Dt} = ((\boldsymbol{v}_\sigma^\alpha + \tilde{\boldsymbol{v}}_\sigma^\alpha)') \circ (\boldsymbol{w}_\sigma^\alpha + \tilde{\boldsymbol{w}}_\sigma^\alpha) \qquad \text{(B.10)}$$

at $\boldsymbol{x} = \boldsymbol{r}_\alpha$.

We will not show this elaboration here and refer to [283] for further details.

Appendix C

Derivation of the Vorton Equations

From the soft-vorton displacement equation (B.9) presented in Appendix B, we can easily derive the vorton displacement equation by taking $\sigma \to 0$ and applying (B.6). The terms $\beta = \alpha$ disappear in a natural manner since for these $g_\sigma = 0$. This means that "self-displacement" of a vorton is omitted [1]. The full equation is given by (8.15) in §8.3.

In first instance, one would be inclined to derive the vorton deformation equation in the same manner as we did for the vorton displacement equation, i.e. from the soft-vorton deformation equation. However, we will take a more direct route and apply the vorton fields directly to the Helmholtz equation. However, because of the delta-functions involved, we have to resort to a technique (mentioned in §8.3) by which the Helmholtz equation will be integrated about the sphere B_α of radius ϵ and centre $\boldsymbol{r}_\alpha$. It is assumed that ϵ is so small that no other vortons are inside the sphere. We call this a weak formulation [2].

For convenience, we split the fields into two parts as described in §8.3. Thus, we have to calculate:

$$\int_{B_\alpha} \frac{D(\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha)}{Dt} = \int_{B_\alpha} ((\boldsymbol{v}^\alpha + \tilde{\boldsymbol{v}}^\alpha)') \circ (\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha) \,. \tag{C.1}$$

The lefthand side of (C.1) can be rewritten as:

$$\int_{B_\alpha} \frac{D(\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha)}{Dt} = \int_{B_\alpha} \frac{D\boldsymbol{w}^\alpha}{Dt} = \frac{d}{dt} \int_{B_\alpha} \boldsymbol{w}^\alpha \,.$$

The last integral is not equal to $\boldsymbol{\gamma}_\alpha$, as one might expect. Since $\boldsymbol{w}^\alpha = \boldsymbol{\nabla} \times \boldsymbol{v}^\alpha$, we get:

$$\begin{aligned}
\int_{B_\alpha} \boldsymbol{w}^\alpha &= \int_{B_\alpha} \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi(\boldsymbol{R}_\alpha) \times \boldsymbol{\gamma}_\alpha) \\
&= -\frac{1}{4\pi\epsilon^2} \int_{\partial B_\alpha} \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\gamma}_\alpha) \\
&= -\frac{1}{4\pi\epsilon^2} \int_{\partial B_\alpha} \{(\boldsymbol{n} \cdot \boldsymbol{\gamma}_\alpha)\boldsymbol{n} - (\boldsymbol{n} \cdot \boldsymbol{n})\boldsymbol{\gamma}_\alpha\} \\
&= -\frac{1}{4\pi\epsilon^3} \int_{B_\alpha} \boldsymbol{\nabla}(\boldsymbol{R}_\alpha \cdot \boldsymbol{\gamma}_\alpha) + \frac{1}{4\pi\epsilon^2} 4\pi\epsilon^2 \boldsymbol{\gamma}_\alpha \\
&= -\frac{1}{4\pi\epsilon^3} \frac{4}{3} \pi\epsilon^3 \boldsymbol{\gamma}_\alpha + \boldsymbol{\gamma}_\alpha
\end{aligned}$$

[1] The exclusion of the term $\beta = \alpha$ can be compared to the so-called cut-off of the kernel of the rule of Biot-Savart (2.3) (see §7.1). A cut-off corresponds, physically, to a finite core size. This suggests that the shortest distance between vortons can be regarded as a core size, a result which has become evident in §10.1 and especially in §10.3.

[2] In our derivation, it is implicitly assumed that the sphere B_α only contains vorton α. Consequently, we require $R_{\alpha\beta} > \epsilon$ for all $\beta \neq \alpha$. On the other hand, the value of ϵ is assumed $\ll 1$. This means that in case of very close approach of vortons the equations will not be reliable. Fortunately, the simulation presented in §10.3 suggests that close approach will not occur.

$$= \frac{2}{3}\boldsymbol{\gamma}_\alpha$$

where $\boldsymbol{n} \equiv \boldsymbol{R}_\alpha / R_\alpha$ and ∂B_α is the surface of sphere B_α. Note that this result is independent of ϵ.

Comparing this result with expression (8.11), we derive the equality [3]:

$$\int_{B_\alpha} \phi''(\boldsymbol{R}_\alpha) \circ \boldsymbol{a} = -\frac{1}{3}\boldsymbol{a} \tag{C.2}$$

for any $\boldsymbol{a} \neq \boldsymbol{a}(\boldsymbol{x})$. Apparently, the second (nonlocal) part of the vorton vorticity field reduces the total amount of vorticity inside sphere B_α with a factor of one third.

Next, we will investigate the four parts of the righthand side of (C.1) separately. Use will be made of the following relation:

$$((\boldsymbol{a} \times \boldsymbol{\nabla}\phi(\boldsymbol{x}))') \circ \boldsymbol{b} = \boldsymbol{a} \times (\phi''(\boldsymbol{x}) \circ \boldsymbol{b}) \tag{C.3}$$

for any $\boldsymbol{a} \neq \boldsymbol{a}(\boldsymbol{x})$ and $\boldsymbol{b} \neq \boldsymbol{b}(\boldsymbol{x})$.

1. $\int ((\boldsymbol{v}^\alpha)') \circ \boldsymbol{w}^\alpha$ represents the "self-deformation" of vorton α.

   Like self-displacement, this part is omitted. This could be justified, as we did in the derivation of the vorton displacement equation above, by the fact that the all components of matrix $((\boldsymbol{v}_\sigma^\alpha)')$ equal zero for $\boldsymbol{x} = \boldsymbol{r}_\alpha$.

2. $\int ((\tilde{\boldsymbol{v}}^\alpha)') \circ \boldsymbol{w}^\alpha$ represents the deformation of the vorticity field $\boldsymbol{w}^\alpha$ generated by vorton α itself, due to the velocity field $\tilde{\boldsymbol{v}}^\alpha$ generated by all other vortons at the location of vorton α.

   Rewriting this integral as

   $$((\tilde{\boldsymbol{v}}^\alpha)') \circ \int \boldsymbol{w}^\alpha ,$$

   we get, by applying (C.3):

   $$-\sum_{\beta \neq \alpha} \boldsymbol{\gamma}_\beta \times (\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \frac{2}{3}\boldsymbol{\gamma}_\alpha) .$$

3. $\int ((\boldsymbol{v}^\alpha)') \circ \tilde{\boldsymbol{w}}^\alpha$ represents the deformation of the vorticity field $\tilde{\boldsymbol{w}}^\alpha$ generated by all vortons except α, due to the velocity field $\boldsymbol{v}^\alpha$ generated by vorton α at the location of vorton α.

   Applying both (C.2) and (C.3), and rewriting this integral as

   $$((\boldsymbol{v}^\alpha)') \circ \int \tilde{\boldsymbol{w}}^\alpha ,$$

   we get:

   $$-\sum_{\beta \neq \alpha} \boldsymbol{\gamma}_\alpha \times [(\phi''(\boldsymbol{R}_{\alpha\beta}) \circ (-\frac{1}{3}\boldsymbol{\gamma}_\beta)] .$$

[3] Here ϕ is the function defined in (8.9) and definitions (8.12) and (8.13) have been applied.

4. $\int ((\tilde{\boldsymbol{v}}^\alpha)') \circ \tilde{\boldsymbol{w}}^\alpha$ represents the deformation of the vorticity field $\tilde{\boldsymbol{w}}^\alpha$ due to the velocity field $\tilde{\boldsymbol{v}}^\alpha$ at the location of vorton α.

   Both fields $\tilde{\boldsymbol{v}}^\alpha$ and $\tilde{\boldsymbol{w}}^\alpha$ are continuous at location $\boldsymbol{r}_\alpha$. Therefore, by the mean value theorem, integration leads to an expression of order ϵ^3, where ϵ is the radius of sphere β_α. Hence, it can be disregarded since $\epsilon \ll 1$.

Taking together all the contributing parts, we get the vorton deformation equation:

$$\dot{\boldsymbol{\gamma}}_\alpha = \sum_{\beta \neq \alpha} \{-\boldsymbol{\gamma}_\beta \times (\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\alpha) + \frac{1}{2}\boldsymbol{\gamma}_\alpha \times (\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\beta)\}$$

or, in full:

$$\dot{\boldsymbol{\gamma}}_\alpha = \frac{3}{4\pi} \sum_{\beta \neq \alpha} \{\frac{1}{2}\frac{\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha}{R^3_{\alpha\beta}} + \frac{(\boldsymbol{R}_{\alpha\beta} \times \boldsymbol{\gamma}_\beta)(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_{\alpha\beta})}{R^5_{\alpha\beta}} + \frac{1}{2}\frac{(\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_{\alpha\beta})(\boldsymbol{\gamma}_\beta \cdot \boldsymbol{R}_{\alpha\beta})}{R^5_{\alpha\beta}}\} . \tag{C.4}$$

In the same way, starting from the transposed Helmholtz equation (8.3) and making use of a rule similar to (C.3), i.e.:

$$((\boldsymbol{a} \times \boldsymbol{\nabla}\phi(\boldsymbol{x}))')^* \circ \boldsymbol{b} = \phi''(\boldsymbol{x}) \circ (\boldsymbol{b} \times \boldsymbol{a}) ,$$

we derive:

$$\dot{\boldsymbol{\gamma}}_\alpha = \sum_{\beta \neq \alpha} \{(\phi''(\boldsymbol{R}_{\alpha\beta}) \circ (\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha) - \frac{1}{2}\boldsymbol{\gamma}_\alpha \times ((\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\beta)\} \tag{C.5}$$

or, in full:

$$\dot{\boldsymbol{\gamma}}_\alpha = \frac{3}{4\pi} \sum_{\beta \neq \alpha} \{-\frac{1}{2}\frac{\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha}{R^3_{\alpha\beta}} + \frac{\boldsymbol{R}_{\alpha\beta}[\boldsymbol{R}_{\alpha\beta} \cdot (\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha)]}{R^5_{\alpha\beta}} - \frac{1}{2}\frac{(\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_{\alpha\beta})(\boldsymbol{\gamma}_\beta \cdot \boldsymbol{R}_{\alpha\beta})}{R^5_{\alpha\beta}}\} . \tag{C.6}$$

Symbols

a	=	vortex ring core radius (fig.2.3)
$\tilde{a}$	=	non-dimensionalized vortex ring core radius (10.1)
$\boldsymbol{A}$	=	vector potential (8.8)
B_α	=	volume of radius ϵ around vorton location $\boldsymbol{r}_\alpha$ (8.16)
E	=	total kinetic energy (9.9)
$E(k)$	=	energy spectrum (9.11)
E_i	=	interaction energy (9.10)
$E_i(k)$	=	interaction energy spectrum (9.13)
E_0	=	self-energy (9.14)
$E_0(k)$	=	self-energy spectrum (9.12)
H	=	total helicity (9.15)
H_i	=	interaction helicity (9.16)
$\boldsymbol{J}$	=	total angular momentum (9.6)
k	=	wave-number
$\boldsymbol{n}$	=	outward normal unit vector
N	=	number of vortons in a vorton ring
$\boldsymbol{P}$	=	total linear momentum (9.2)
$\boldsymbol{r}_\alpha$	=	location vector of a vorton labelled α
R	=	vortex ring radius (fig.2.3)
$\boldsymbol{R}_\alpha$	$\equiv$	$\boldsymbol{x} - \boldsymbol{r}_\alpha$
$\boldsymbol{R}_{\alpha\beta}$	$\equiv$	$\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta$
Re	=	Reynolds number (§A.3 of Interlude)
t	=	time
$u(y)$	=	shear flow velocity profile (10.8)
U	=	outer shear flow velocity (10.8)
$\boldsymbol{v}$	=	velocity
V	=	vortex ring velocity (fig.2.3)
$\tilde{V}$	=	non-dimensionalized vortex ring velocity (10.5)
$\boldsymbol{w}$	=	vorticity (1.1)
$\bar{\boldsymbol{w}}$	=	diagnostic vorticity (9.18)
$\boldsymbol{x}$	=	spatial location
x, y, z	=	components of $\boldsymbol{x}$
$\boldsymbol{X}$	=	material location
X_1, X_2, X_3	=	components of $\boldsymbol{X}$
α, β	:	labels of vortons
$\boldsymbol{\gamma}_\alpha$	=	strength vector of a vorton labelled α
Γ	=	circulation (4.2)
δ	=	shear flow height (10.8)
$\phi(\boldsymbol{x})$	$\equiv$	$1/(4\pi x)$ (8.9)
$\boldsymbol{\omega}$	=	angular velocity (2.1)

$\boldsymbol{\nabla}$	=	spatial nabla operator (1.1)
$\frac{\partial}{\partial t}$	=	spatial derivative
$D\ /Dt$	=	material derivative (1.3)
$\delta(...)$	=	Dirac delta function
$\cdot$	=	scalar product
$\times$	=	vector product
$(\boldsymbol{v}')$	:	deformation matrix (8.2)
$(\boldsymbol{v}')^*$	:	transposed of matrix $(\boldsymbol{v}')$
$\int_V$	:	volume integral
$\int_{\partial V}$	:	surface integral
$\oint_C$	:	contour integral

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Samenvatting van *Over Wervelatomen en Vortonen*

Dit proefschrift bestaat uit twee delen. In het eerste wordt de ontwikkeling van een 19e-eeuws atoommodel, het wervelatoom (*vortex atom*), behandeld. In het tweede deel wordt de recent geïntroduceerde vortonmethode besproken en de numerieke simulaties die hiermee zijn uitgevoerd om haar te testen. In een zgn. *Interlude* worden enige ontwikkelingen in de stromingsleer geschetst uit de tussenliggende tijd die belangrijk zijn voor het begrijpen van onderdelen van het tweede deel.

De ontwikkeling van het wervelatoom kan alleen begrepen worden als men enig zicht heeft op de ontwikkelingen op het gebied van materietheorieën en van het begrip vorticiteit binnen de stromingsleer. Op het moment dat het wervelatoom in 1867 door Kelvin werd geïntroduceerd bestond vorticiteit en de bijbehorende theorie nog pas enkele decennia. Hoewel geleerden als Cauchy en Stokes rond 1845 al enige bijdragen hadden geleverd, legde pas in 1858 Helmholtz de basis van de vorticiteitstheorie. Aan hem danken we enige belangrijke definities, theorema's, vergelijkingen en fysische inzichten. Op het gebied van materietheorieën moeten we het klassieke atoom van Democritus noemen. Het atoom als hard, onveranderlijk bolletje was ook in de 19e eeuw nog populair, met name dankzij de opkomst van de kinetische-gastheorie.

De ontwikkeling van Kelvins ideeën op het gebied van de zgn. ether, de hydrodynamica en het electro-magnetisme, samen met de invloed van Faraday, Rankine (het *molecular vortex*) en Stokes, zijn belangrijk voor een goed begrip van de introductie van het wervelatoom door Kelvin. Daarnaast moet als directe aanleiding Taits experiment met rookringen genoemd worden. De elastische wervelringen, zo meende Kelvin, konden veel beter de diverse eigenschappen van materie beschrijven dan het Luctrius atoom: onverwoestbaarheid, zwaartekracht en inertie, spectra.

Kelvins model werd echter over het algemeen koel ontvangen, zeker op het Continent, waar men vooral filosofische bezwaren tegen Kelvins manier van modelvorming had. In Groot-Britannië probeerden enkele aanhangers langs analytische weg aan te tonen dat het wervelatoom inderdaad belangijke voordelen bood, maar slaagden hierin nauwelijks. Daarbij werden zij vooral gehinderd door gebrek aan wiskundige technieken. Intussen groeide het aantal pogingen aan te tonen dat het model niet bruikbaar was en dat de beweringen van aanhangers obscuur waren. Ook aanpassingen van Kelvins oorspronkelijke atoom leverden niets op.

Pogingen om het wervelatoommodel toe te passen in ethermodellen liepen ook op niets uit en Kelvin verloor het geloof in zijn eigen schepping. Zo kwam hij onder andere tot het inzicht dat wervelringen niet altijd stabiel hoefden te zijn. De ontdekking van het electron en het inzicht dat electrische lading fundamenteel voor een atoommodel was, gaf Kelvins model de genadeslag. Daarnaast was Kelvins manier van modelvorming uit de mode geraakt.

De wervelatoomtheorie, hoe onvruchtbaar verder ook voor de atoomtheorie, heeft een belangrijke stimulans betekend voor het onderzoek naar vorticiteit en wervelstructuren. Zo zijn inmiddels vele, meer geraffineerde, wervelringexperimenten uitgevoerd, de analytische uitwerking is verder ontwikkeld en met name m.b.t. de stabiliteit van wervelringen werden opmerkelijke resultaten gevonden (Kelvins model van de wervelring bleek niet stabiel). Ook werd vorticiteit een vertrouwd begrip in het onderzoek naar turbulente stromingen, waar een van de belangrijkste ontwikkelingen de ontdekking van coherente (wervel)structuren is. Daarnaast hebben in de moderne stromingsleer begrippen als heliciteit en wervelreconnectie zich een belangrijke

plaats verworven in de zgn. *topological fluid mechanics.*

Een andere belangrijke moderne ontwikkeling is de opkomst van het gebruik van de computer: "computational fluid mechanics". Daarbinnen valt de opkomst van de zgn. wervelmethoden (*vortex methods*): het numeriek simuleren van wervelstructuren d.m.v. modellering met "wervelelementen" (bijv. *vortex-filaments* en *vortex-points*). We moeten echter wel enige eisen opleggen aan deze methoden: een divergentievrij vorticiteitsveld, correcte modellering van de vorticiteitsdistributie, correcte modellering van de deformatie en interactie van wervelstructuren, behoud van bewegingsinvarianten, geen negatieve gevolgen van *remeshing*, correcte oplegging van randvoorwaarden, convergentie, en aanvaardbare rekeninspanningen. Eén van de recente wervelmethoden is de vortonmethode, het onderwerp van het tweede deel van dit proefschrift.

Het wervelelement toegepast in de vortonmethode, is de vorton Dit kunnen we opvatten als een driedimensionale puntwervel. De deformatie van deze vortons volgt uit de zgn. Helmholtz vergelijking. De deformatie- en de verplaatsingsvergelijking voor de vortonen worden vervolgens numeriek opgelost. Er is echter over de afleiding van de deformatievergelijking discussie ontstaan in de literatuur. Uitgaande van twee in principe gelijkwaardige vormen van de Helmholtzvergelijking kwamen Novikov en Kuwabara tot twee niet-gelijkwaardige vortondeformatievergelijkingen. In dit proefschrift stellen wij een nieuwe afleiding van de vergelijking voor, die deze inconsistentie opheft en aantoont dat Novikovs noch Kuwabara's vergelijking aantrekkelijk is. Eén van de doelen van onze numerieke simulaties is geweest om de superioriteit van onze vergelijking aan te tonen; wij menen dat dit is gelukt; Kuwabara's vergelijking blijkt in elk geval onbruikbaar. Een ander doel van de simulaties is geweest het vergelijken van het gedrag van diverse wervelstructuren met dat van hun vorton-equivalenten. Wij hebben ons beperkt tot onderzoek naar het gedrag en de interactie van vortonringen, het vorton-equivalent van de wervelring. We waren uiteraard afhankelijk van de beschikbaarheid van experimentele, numerieke en analytische resultaten van anderen en van de mogelijkheden m.b.t. randvoorwaarden (alleen een *free-slip*-voorwaarde is mogelijk). Verder hebben we de volgende wervelfenomenen willen simuleren: werveldeformatie, wervelkerndeformatie, wervelreconnectie, en *alignment* van wervelbuizen. De configuraties die we hebben gesimuleerd zijn: een enkele vortonring (onderzoek naar kern en stabiliteit); een enkele pseudo-elliptische vortonring (deformatie en reconnectie); de botsing van twee coaxiale vortonringen (kerndeformatie en stabiliteit); de interactie van twee aanvankelijk parallel bewegende vortonringen (reconnectie); de interactie van twee "geknoopte" vortonringen (*alignment*); en een enkele vortonring in een afschuifstroming boven een vlakke plaat (een mogelijk eenvoudig model voor het gedrag van coherente structuren in een turbulente grenslaag).

Ook is de zgn. *vorton-division*-techniek onderzocht: het toevoegen van vortonen op plaatsen waar de afstand tussen naburige vortonen groter wordt dan een bepaalde waarde. De simulaties hebben laten zien dat in elk geval toevoegen zonder *updating* van de vortonen niet aanvaardbaar is. Met *updating* kan enige verbetering optreden, maar *division* met lineaire interpolatie blijkt geen remedie tegen ontsporende simulaties.

De simulaties hebben in elk geval laten zien dat aan de vortonring een kern(diameter) kan worden toegekend. De verdeling van de vorticiteit is echter niet homogeen verdeeld over de ring. Het aantal vortonen in de ring bepaalt de kerndiameter (en snelheid), zodat numerieke nauwkeurigheid niet met het aantal vortonen verbeterd kan worden (dit zou wellicht wel kunnen door een adnere modellering van de wervelring). Het feit dat de kerndiameter groeit bij wervelstrekking lijkt fysisch gezien onacceptabel. De discrete representatie van con-

tinue wervelstructuren is enerzijds de oorzaak van de mogelijkheid reconnectie te simuleren (bij dichte nadering van vortonen treedt een "bifurcatie" op naar een nieuwe stabiele situatie; de vraag is hoe "fysisch" dit is), aan de andere kant leidt het tot chaotisch gedrag van de vortonen bij dichte nadering. *Alignment* wordt goed gerepresenteerd, maar ook hier lopen de simulaties mis bij dichte nadering van de vortonen; in experimenten treedt waarschijnlijk annihilatie van vorticiteit op.

De toepassing van de vortonmethode bij het onderzoek naar coherente structuren in een grenslaag wordt gehinderd door de mogelijk belangrijke invloed van de *no-slip*-conditie aan de wand. Toch laat onze eenvoudige simulatie interessant gedrag van de vortonring zien en een grote piek in een grootheid die verband houd met de *Reynolds shear stress*. Dit duidt op het optreden van een zgn. *burst*; het optreden daarvan wordt in elk geval bepaald door de grootte van de snelheid buiten de grenslaag.

We kunnen concluderen dat de vortonmethode relatief eenvoudig is, weinig rekeninspanning vergt, en in bepaalde situaties goede simulatieresultaten oplevert. Wij moeten echter niet tveel van de methode verwachten, enerzijds vanwege de discrete representatie, anderzijds vanwege het ontbreken van visceus gedrag.

In de *Epilogue* worden tenslotte beide delen van het proefschrift weer bij elkaar gebracht opeen wetenschapsfilosofisch niveau. Zowel Kelvin bij de ontwikkeling van zijn wervelatoomtheorie als huidige onderzoekers die simulaties uitvoeren met *vortex methods* als de vortonmethode, stuiten op de beperkingen of onvolkomenheden van hun zgn. *computational model*. Samen met andere factoren was dit de oorzaak van de beperkte bloei van Kelvins atoomtheorie. Wat betreft onze simulatie van de vortonring in een *shear flow* boven een plaat: hierbij hebben we niet alleen te maken met een wellicht inadequaat *computational model*, maar we moeten ons ook afvragen of deze modellering van turbulente grenslaagstromingen zinvol en niet misleidend is.

Levensloop

Ik werd geboren op 30 juli 1966 in Gouda en bracht daar mijn eerste levensjaren door. In de periode 1978-1984 bezocht ik het Fioretti College in Lisse en behaalde daar het diploma Gymnasium-β. Vervolgens studeerde ik in de periode 1984-1989 Werktuigbouwkunde aan de Technische Universiteit Delft. Na mijn afstuderen werd ik enige maanden door het Delfts Universiteitsfonds betaald tot een officiële aanstelling volgde als Onderzoeker in Opleiding in dienst van de Stichting FOM (FOM-werkgroep SWD-b). Mijn onderzoek vond plaats bij de vakgroep Stromingsleer van de Faculteit der Werktuigbouwkunde en Maritieme Techniek (Laboratorium voor Aero- en Hydrodynamica).