## Stellingen bij het proefschrift van A.J.Q. Alkemade

1. Het leggen van verbanden tussen passages uit Lucretius' *De Rerum Natura* en de turbulentietheorie zoals gedaan door Serres is onzinnig.

   M. Serres, *La naissance de la physique dans le texte de Lucrèce; fleuves et turbulence.* Paris, 1977.

2. Een gebrek aan historische kennis bij beoefenaren van een bepaalde tak van wetenschap uit zich in de naamgeving van vergelijkingen en theorema's.

3. Bij het schrijven van een numeriek programma moet men niet alleen letten op de convergentie van het rekenschema zelf, maar ook op de convergentie van het schrijven van dit programma.

4. De pogingen van Lumley en Kline om een relatie te leggen tussen Kuhns opvattingen over wetenschappelijke revoluties en de huidige crisis in het turbulentie-onderzoek zijn zwak en houden ten onrechte geen rekening met de kritiek zoals deze bijvoorbeeld door Laudan op Kuhns ideeën zijn geuit.

   J.L. Lumley (ed.), *Whither Turbulence? Turbulence at the Crossroads.* Berlin, etc., 1990.

   L. Laudan, *Progress and its Problems; Towards a Theory of Scientific Growth.* Berkeley, etc., 1977.

5. Er is geen goede rechtvaardiging te vinden voor het gebruik van een Gaussische verdeling van de vorticiteit in de kern van een wervelring bij modellering daarvan.

6. Het schrijven van een historisch overzicht van de ontwikkeling van het onderwerp waarop men promoveert is geen verspilling van tijd en moeite.

7. Ondanks een eerbiedwaarige geschiedenis is het verschijnsel 'wervel' (*vortex*) nog altijd onvoldoende eenduidig gedefinieerd in de wetenschappelijke literatuur. Deze situatie hindert de vooruitgang van het onderzoek op het gebied van werveldynamica en haar raakvlakken met turbulente stromingen.

8. De invoering van het begrip *inviscid dissipation* van Aksman *et al.* draagt niet bij tot een verklaring voor het niet behouden zijn van de zgn. interactie-energie bij numerieke simulaties met vortonen.

   M.J. Aksman, E.A. Novikov, S.A. Orszag, "Vorton method in three-dimensional hydrodynamics". *Phys. Rev. Lett.* 55 (1985) 2510.

9. Het bestaan van onderzoekscholen op een vakgebied is een noodzakelijke noch voldoende voorwaarde voor 'toponderzoek'.

10. Het is niet goed dat men in discussies bij voorbaat of zonder goede reden van zijn overtuiging afwijkt of deze niet naar voren brengt. Het is echter ook niet goed, zoals Burgers lijkt te stellen, dat men in zijn meningen geen enkele toegeeflijkheid betuigt jegens andersdenkenden.

    J.M. Burgers, *Het Atoommodel van Rutherford-Bohr.* Proefschrift. Haarlem, 1918.

# On Vortex Atoms and Vortons

**Proefschrift**

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus,
prof.ir. K.F. Wakker,
in het openbaar te verdedigen
ten overstaan van een commissie
aangewezen door het College van Dekanen
op dinsdag 5 april 1994 om 16.00 uur
door

**Alfons Johannes Quirinus Alkemade,**

werktuigkundig ingenieur,
geboren te Gouda.

Dit proefschrift is goedgekeurd door de promotoren:

- prof.dr.ir. E.W.C. van Groesen
- prof.dr.ir. F.T.M. Nieuwstadt
- prof.dr. H.A.M. Snelders



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Aan haar zonder wier ...

*Toute l'âme résumée*
*Quand lente nous l'expirons*
*Dans plusieurs ronds de fumée*
*Abolis en autres ronds*

*Atteste quelque cigare*
*Brûlant savamment pour peu*
*Que la cendre se sépare*
*De son clair baiser de feu*

*Ainsi le choeur des romances*
*A ta lèvre vole-t-il*
*Exclus-en si tu commences*
*Le réel parce que vil*

*Le sens trop précis rature*
*Ta vague littérature*

Stéphane Mallarmé

**Het promotie-onderzoek waarvan dit proefschrift de weerslag vormt, werd mogelijk gemaakt door financiële ondersteuning vanuit het Stimuleringsgebied Stroming en Warmte van de Stichting voor Fundamenteel Onderzoek der Materie. De auteur is FOM hiervoor zeer erkentelijk.**

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# Contents

# Prologue

*The history of the development, whether normal or abnormal, of ideas is of all subjects that in which we, as thinking men, take the deepest interest.*
J. Clerk Maxwell

This thesis deals with two topics which are related to the concept of vorticity. Therefore, it consists of two parts. The "vortex-atom-part" shows the development of a theory of matter, introduced by the English scientist Lord Kelvin in 1867, which would attract the attention of several 19th century scientists up to the beginning of our century. Kelvin's "vortex atom theory" can be put into the context of several developments in 19th century physics, especially those with regard to theories of matter and the still developing theory of rotational flow or vorticity. Therefore, the vortex-atom-part not only tries to sketch the actual development of the vortex atom itself, it also provides a historical background for this development.

The second part, the "vorton-part", is an account of the theoretical foundation and the application to numerical simulations of the vorton method. This is one of the many vortex methods, applied nowadays to the (numerical) study of flow phenomena. Vortex methods are based on the fact that vortices play important roles in fluid flows and can be regarded as important applications of the knowledge on vortex motion which has been gathered in the past centuries and of the surging use of numerical techniques in fluid mechanics. The vorton method will be investigated by means of numerical simulation of several test cases. Most of these were already studied by the scientists who occupied themselves with the elaboration of the vortex atom model or who were just incited to research on vortex motion by this model. However, their investigations were largely hindered by mathematical difficulties. Today, the use of vortex methods as computational tools may provide more insight into the kinematics and dynamics of vortex structures.

In an "interlude" between these two parts, I will try to indicate how several of the issues which will be regarded in the vorton-part have a past going back to the period of the flowering of the vortex atom (and even to earlier times) [1]. This part of the thesis may be of interest to the historically interested reader who wants to become informed of results from research of more than a century ago still have a bearing on modern research. It may also be of interest for the "physical" reader [2], who wants to know about the roots of concepts he is using today. In the Interlude I will also take the opportunity to introduce some of the recent and essential developments in vorticity theory and how several topics related to vorticity and vortex motion play important roles in modern fluid mechanics, including turbulence.

In the Epilogue I will look back on the two parts of this thesis from a broader point of view, i.e. on the development of scientific theories and the use of models. Especially, attention

[1] Nice illustrations of the continuing traditions in vortex dynamics can be found in Saffman's recent book [205].

[2] I.e. the reader interested in current scientifically obtained results, e.g. those presented in the vorton-part of this thesis.

is given to the possible lessons that one may learn from the development of the vortex atom with regard to the present modelling of turbulent flows.

During the research on both parts, I became convinced that their combination was more interesting than I had initially realized. In the field of fluid mechanics, and probably in many others, the appreciation of history is still largely absent or at most only commencing. I also realized that it would be nonsensical to try to show any direct relationships between the two parts. However, it should be possible to let the "physical" reader realize that modern science has a past and, more importantly, is influenced by this past. Still later, I started to realize that the development of the vortex atom might even teach modern scientists the use of models. Both parts may be read separately but I have taken the opportunity to refer in the vorton-part to equations and theorems which have been introduced in the vortex-atom-part.

The writing of both parts has required a different approach. In an historical account the writer has to be careful in studying and treating the original research, which, consciously or subconsciously, he is constantly comparing to present knowledge. He should try to treat the developments which he describes from the point of the view which scientists had at the time corcerned. Therefore, he has to take account of the *status quo* of science during the time he is writing about, and usually also of the developments which led to this state.

In a "physical" account of a theory, the writer is usually not bothered with the origin of the results he applies. First of all, he may not be aware of the historical development, secondly his striving for clarity and a logical presentation of the different parts of his story does not allow treatment of the sometimes confusing and obscure historical developments of the concepts that he uses.

The style of writing is also be different between the two parts. As a historian, an author is telling a story. He tries to indicate how the subject of his story could arise, how it has developed, and, as in the case of the vortex atom, how it declined. In the treatment of his subject he is also trying to convince the reader of some issue, e.g. of his opinion on the reasons why a decline could happen. This view on the philosophy of science is implicitly valid in the story he writes and the choices he makes. For him, the story is not just an historical account.

As a physicist, an author is trying to show whether a method or theory or, more generally, an approach is suitable or unsuitable for certain applications. Usually, he gives some background and details on the approach itself, but the most important part of his presentation are the conclusions, in which he may refute other approaches and give suggestions for further research. The "physical author" will (and should) try to provide the facts and to draw his conclusions objectively; he does not intent to present a story, but an account.

## Prologue to the Vortex-Atom-Part

The development of a scientific concept is the result of both scientific discussion *and* of personal circumstances, friendships, quarrels. A researcher writing on some specific area of the history of science has to investigate all relevant information: not only the material published in journals and books, but also the, mostly unpublished, letters, notebooks, etc. For the writing of the history of the vortex atom, I have only investigated the published material that I could find and consult, and, for reasons of time and financial support, made no attempt to search in the archives; therefore, I will not claim that this story is complete by any means [3]. However, I am

[3] I could trace only two papers dealing especially with the vortex atom: the paper by Silliman [215] and the M.A. Thesis by Pauly [173].

convinced that the most relevant information is presented here. In the presentation, I have tried to take care that the story doesn't become a summation of facts and opinions. In that case, the reader would soon find himself lost and the "physical" reader would be confirmed in his opinion on the relevance of historical surveys.

The vortex-atom-part tells the story of the rise, elaboration, and fall of a scientific theory (or model; see the Epilogue). This story consists of six chapters of which the first three chapters can be regarded as an exposition of the prehistory of the vortex atom and the road towards its birth. In Chapter 1 the development of fluid mechanics will be shortly described, and more specifically the rise of the concept of vorticity and its theory. This chapter ends with the appearance, in 1858, of Helmholtz's fundamental paper on vorticity theory which, because of its special importance in the development of the theory, is treated separately in Chapter 2. To understand the introduction of the vortex atom by Kelvin, in Chapter 3 I roughly sketch his scientific development. Besides, this chapter contains an account of the direct incentive which led to Kelvin's proposal of the vortex atom, i.e. Tait's experiment with smoke rings. In 1867 Kelvin not only introduced the vortex atom in his paper "On Vortex Atoms", he also started work on "vortex motion" in general, leading to a seminal paper in 1869. Both will be discussed in Chapter 4. In Chapter 5, I will treat the reception and development of the vortex atom theory, both in and outside Britain. Not only fundamental criticism started to rise, elaboration of the model also showed its inability to comply with several essential requirements related to its status as a theory of matter. In Chapter 6 the decline of the vortex atom is treated. Despite the general awareness of its weakness, the theory had drawn the attention of scientists building models of the ether, one of the most haunting riddles in physics at that time. However, the vortex ethers appeared as difficult to elaborate as the vortex atom model and they couldn't solve this riddle. Meanwhile, even Kelvin himself had lost faith in the vortex atom. The rising consciousness of the importance of "electricity" in the atom, accompanied by the discovery of the electron, gave the vortex atom its final deathblow.

### Prologue to the Vorton-Part

The study of vorticity and vortical flows has remained a constant topic of active research in fluid mechanics after the fall of the vortex atom. Though a direct correlation may be hard to prove, the last few decades have seen an extra stimulus in research on vortex motion, due to the growing interest in the role of so-called coherent structures in turbulent fluid flows. These structures are generally thought to be of vortical nature and understanding of their behaviour and interaction may be essential to a solution of the turbulence problem.

Though nowadays it seems that turbulent flows can be simulated numerically by completely solving the governing equations, available computational power still restricts the range of turbulent flows (e.g. expressed by means of the so-called Reynolds number). Vortex methods, tools to simulate only the vortical parts of flows, may be a means to circumvent these restrictions and, as a result, may increase our insight into turbulence. If we may represent a coherent structure as an elementary vortex configuration, e.g. as a vortex ring, vortex methods are the ideal means to study its behaviour. In a full "phenomenological modelling" of e.g. turbulent boundary layers one may then restrict the simulation to one or a few of these vortical structures.

Chapter 7 contains a general review of vortex methods. One of the vortex methods which have been introduced during the last few decades is the vorton method, whose merits and capabilities are still relatively unexplored. In Chapter 8 the general characteristics of the vorton method will be presented. The vorton fields and the equations describing the displacement and deformation of vortons will be derived. Application of the vorton method means the representation of continuous vortex structures by the discrete vortons and the numerical solution of the vorton equations. This gives the behaviour of the vortex structures.

To gain insight into the possibilities and limitations of (the application of) the vorton method, I have simulated six basic configurations or test cases, all of them involving the vorton equivalent of the vortex ring, i.e. the vorton ring. In Chapter 9 I defend the choice of these cases and present the diagnostics used in their investigation. In Chapter 10, for each of the six test cases the vorton simulation results are presented in combination with recent results from literature. However, experimental data are scarce, especially quantitative data. This has somewhat limited the possibility to investigate the value of the vorton method presented here. Nevertheless, in the general discussion of the simulation results in Chapter 11, I have been able to formulate several conclusions; besides, some suggestions for future research are put forward.

Special attention is drawn to §10.6. The original aim of my work has been to regard the question: "can the vorton method be applied to the study of coherent structures in the turbulent boundary layer?". This layer is characterized by a shear flow profile and a no-slip boundary condition at the wall. Since indications exist which point out an active role of vortex rings in boundary layers, in §10.6.2 a vortex ring has been taken as basic ingredient to investigate the above question.

In writing the vorton-part, I have assumed that the reader is familiar with the basics of fluid mechanics, i.e. with the generally used equations and symbols. Besides, he is supposed to be familiar with the results of vorticity theory mentioned in Chapter 2 and §4.2 [4] and of the physical concepts introduced in the Interlude.

Regarding the numerical simulations, I want to remark that I only used the computing facilities available at the Laboratory of Aero- and Hydrodynamics (a HP-minicomputer). Obviously, a larger computer would have fastened the computations or would have made more extensive computations possible. At this stage I found no need to resort to this.

Furthermore, I have not tried to optimize the numerical scheme on purpose such that calculations ran as fast as possible; my attention has principally been devoted to getting a correct vortex method, not the fastest one.

## Further Remarks

- Throughout this thesis, I have tried to use the same symbols for all physical quantities involved. This meant that I had to adapt several of the symbols used in the older literature. The reader could object that this attitude obstructs insight into the development of notation in this part of physics and may deceive the unsuspecting reader, who will believe that regarding symbols fluid mechanics did not develop or reached agreement immediately after introduction. I have chosen for convenience and clearness and can only encourage the reader to read the original literature himself.

  For convenience a list of symbols is provided at the end of this thesis.

- For the same reason, vector notation is generally used for reasons of convenience. Note, however, that in 19th century literature, e.g. the works of Helmholtz and Kelvin, this was not yet common practice.

[4] For a more thorough and mathematical introduction to vorticity theory, consult [115], [268], [210], and [205].

- Unless otherwise stated, the theorems and equations used in this thesis are only valid for:
  - Incompressible flows: flows for which the velocity field satisfies $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, i.e. the velocity field is divergencefree. Actually, most results in this thesis are valid for a wider class of flows, i.e. barotropic flows: flows for which the the pressure is only a function of density $\rho$ and/or time $t$, i.e. $f(p, \rho, t) = 0$ for some function $f$.
  - Inviscid or perfect flows: flows in which viscosity plays no role.
  - Flows under the action of conservative body forces, i.e. forces which can be represented as a gradient of a force potential (e.g. the gravitational force).
- Regarding terminology, I have to remark that the terms vortex motion, vorticity theory, vortex dynamics, and some others are not used according to any strict rules.

# Chapter 1

# Vorticity before 1858

The concept of the vortex atom, the subject of the first part of this thesis, could only arise after the mathematical and physical basis of the theory of vorticity and vortex motion [1] had been laid. This had happened mainly during the second half of the 18th and the first half of the 19th century. The theory became definitely established as a serious branch in fluid mechanics when the German scientist Helmholtz published a fundamental paper in 1858. The development of fluid mechanics itself can be traced back to classical Greece, but has only been treated as a serious part of mechanics since Newton. Since then it has constantly occupied and fascinated many of the most highly esteemed scientists.

In §1.1 a concise survey is given of the development of fluid mechanics up to the middle of the 19th century. It mainly serves as a means of setting vorticity theory in an historical order and background and as a source of references on this development. In §1.2 the history of man's fascination with vortices and the eventual introduction of vorticity into fluid mechanics is treated.

## 1.1 A Short Survey of the Development of Fluid Mechanics

Despite mankind's long-standing interest in fluid mechanics, so far few, if any, serious comprehensive studies on its history have been published. A thorough account of scientific research on fluid motion up to the works of Lagrange was published in an extensive "Prologue" to the volume of Euler's *Opera Omnia* devoted to fluid mechanics [267]. Naturally, one can find accounts of the history of fluid mechanics in works on the history of mechanics in general, such as [50] and [226]. Furthermore, two works on the history on fluid mechanics, though rather global and superficial, have been published: those by Rouse & Ince [203] and by Tokaty [265]. An older "historical sketch" (particularly useful for the development of aerodynamics) can be found in [64]. For a description of the developments in our century, we only have concise surveys like [67].

A table showing essential parts of the development of fluid mechanics up to the Second World War is given by table 1.1. This development is divided in four periods which can roughly be characterized both by fields of research and by the various scientists who have contributed to the development of these fields. In the last column some references are given [2].

[1] Vortex motion is usually called rotational motion nowadays [268, §29]. However, we will keep to this term in honor of Kelvin's paper treated in §4.1.

[2] For general biographical accounts of the scientists mentioned, we refer to the several volumes of the *Dictionary of Scientific Biography* [65].

| | | |
|---|---|---|
| **hydrostatics** | | |
| | ARCHIMEDES (187 B.C.-212 B.C.)<br>STEVIN (1548-1620)<br>PASCAL (1623-1662) | <br>[226]<br>[226] |
| **classical/mechanistic fluid mechanics:** | fluid mechanics as part of (rational) mechanics | |
| - fundamentals | NEWTON (1642-1727)<br>D. BERNOULLI (1700-1782)<br>EULER (1707-1783) | [267] [226]<br><br>[267] [226] |
| - ballistics | EULER | [267] [226] |
| - potential flow theory | D'ALEMBERT (1717-1783)<br>EULER | [267] [226]<br>[267] |
| **mathematical fluid mechanics:** | mathematical treatment of physical flows | |
| - viscous flows | NAVIER (1785-1836)<br>POISSON (1781-1840)<br>DE SAINT-VENANT (1797-1886)<br>STOKES (1819-1903) | [226]<br>[226]<br>[226]<br>[281] |
| - vorticity theory | HELMHOLTZ (1821-1894)<br>KELVIN (1824-1907) | [110]<br>[218] |
| - gas dynamics | RIEMANN (1826-1866)<br>HUGONIOT (1851-1887)<br>DE SAINT-VENANT | [226]<br><br>[226] |
| **modern fluid mechanics:** | towards (engineering) applications | |
| - turbulence | REYNOLDS (1842-1912)<br>PRANDTL (1875-1953)<br>TAYLOR (1886-1975)<br>BURGERS (1895-1981)<br>VON KÁRMÁN (1881-1963) | [150]<br>[202]<br>[20]<br><br>[100] |
| - aerodynamics | LANCHESTER (1868-1946)<br>VON KÁRMÁN<br>ZHUKOVSKY (1847-1921)<br>PRANDTL | [104]<br><br>[223]<br>[202] |
| - rheology | MAXWELL (1831-1879)<br>BURGERS | |
| - geophysical fluid mechanics | RICHARDSON (1881-1953)<br>BJERKNES (1862-1957)<br>ROSSBY (1898-1957) | [17]<br>[55] |

Table 1.1: Survey of the development of fluid mechanics.

Some remarks have to be made about this table. First, the sequence of fields and names is not meant to indicate any rate of importance. Second, the classification of periods and their subdivions into areas of research is an oversimplification and only superficial. The names of the periods and the short explicative slogans are our own interpretations. The representatives mentioned here are generally regarded as some of the most famous ones, but several others should certainly be included in any serious extension of this table. For the references only the most relevant and readily available for each representative have been chosen. Third, one should not get the impression that e.g. the development of hydrostatics ended with the works of Pascal or that today the theory of potential flows has become useless or neglected after the rise of the theory of viscous flows.

Although many entries of this timetable will not be relevant to the understanding of the rise of vorticity theory, they are given here to enable the reader to place the development of our topic in a historical context. In the next section, it will be shown how the parts which *are* important (mainly the insuffiency of potential flow theory), have made their contributions, eventually leading to Helmholtz's results.

## 1.2 History of Vorticity up to 1858

One of the first pronouncements on the role of vortices is by the Greek Democritus (400 B.C.), who based his theory on that of Leucippus. In the 5th century B.C. Leucippus had supposed that the collisions of atoms in random motion would give rise to a vortex. In Diogenes Laertius' account of Democritus's philosophy, we read:

> All things come into being by necessity, the cause of the coming into being of all things being the vortex, which he [Democritus] calls necessity.

The meaning of this statement remains obscure [3] and Democritus is better remembered for his theory of matter, which has furnished him with the name of the Atomist.

For the followers of Democritus, the constitution of matter rested on the existence of an infinite number of indivisible and impenetrable particles. These had weight and hardness but didn't exercise any force on each other. Where there was no "Being" (matter), there was "Not-Being" which could be called vacuum or empty space. Other important characteristics of the Democritean atoms were their invariability (no change of shape) and complete equality in quality. According to the Atomistic viewpoint, the motion of bodies *per se* could be explained in terms of the motion of the atoms. The difference between specific bodies were attributed to the shape, position, configuration and kind of motion of the atoms; hence, mechanical concepts played a fundamental role in this atomic model.

One of the interesting aspects of the Democritean theory concerned the origin of so-called worlds: the atoms have been moving for ever in the infinite, empty space. Through interaction they will form whirling conglomerates, which expand into the worlds, which can be regarded as complexes of atoms. The worlds are born in an infinite number next and after each other, but in course of time they will desintegrate again into their constituents. One of them is our world. However, this process doesn't take place randomly, but it happens by necessity. Maybe, we have to regard Democritus' words given above in this context of the worlds.

[3] According to [224], a rich source of original texts on atoms, the whirling motion is actually a kind of shuffling motion. For a survey of the role of vorticity in Greek antiquity, see the only general work on vortex motion, by Lugt [134, Ch 1].

The word *vortex* in its present meaning probably appeared for the first time in the discussion of meteorological phenomena in *De rerum natura*, a didactic poem by the Roman poet Lucretius (50 B.C.). Lucretius can be seen as a follower of the Democritean tradition, since his poem treats the doctrine of the Greek philosopher Epicurus who in his turn was strongly influenced by Democritus' doctrine of nature.

Besides meteorology, Lucretius discussed matter. Everything was built up by an infinite number of atoms and voids. The atoms were as indivisible, eternal and invariable as the Democritean ones. However, whereas the motion of the Democritean atoms was completely indeterministic, the Epicurean atoms had an additional "degree of freedom" to which Lucretius attached the name of *clinamen* [4]: the atoms' motion along straight lines due to free fall could be disturbed causing a small, spontaneous, deviation from these straight motions and a collision and accumulation of the atoms. By this phenomenon, Lucretius explained the birth of the "All", i.e. of all beings.

Though the fascination with vortices must have been constantly present during the ages, up to Descartes we only have some drawings of whirls as observed by Leonardo da Vinci (1452-1519) [5].

For Descartes (1596-1650) physical science, that is to say his theory of matter and motion, rested on the basic assumption that matter equals extensiveness (*extension*) in space. This led him to reject the concept of *actio in distans*, which by that time had already been a serious point of discussion in the explananation of e.g. the working of gravity. The existence of indivisible parts, the vacuum, and absolute motion also didn't fit in Descartes' picture. Consequently, he rejected the Democritean and Lucretian theory of matter.

We won't go into the details of Descartes's ingeneous doctrines. We just mention his original world system, which he thought able to describe every single motion on earth and in heaven. One of the most intriguing parts of his system was the omnipresence of vortices (*tourbillons*). Without going into any details of this vortex theory [6], we only remark that according to Descartes push, pull, and (vortical) motion of material bodies could explain all phenomena in nature.

Newton's severe criticism of Descartes' doctrines, which had appeared of little heuristic value, initiated a new era in physical science at the end of the 17th century, the beginning of the period in which the use of force and mathematical analysis became dominant. In the 18th century the trend towards mathematization of different aspects of physics was continued. One of its promotors was Euler whose contributions can be found in almost every branch of science.

In fluid mechanics, we owe to Euler the so-called velocity potential which can be traced to the years 1752-1755 [268, §36]. If a velocity field $\boldsymbol{v}$ satisfies the condition

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \mathbf{0} ,$$

it follows that $\boldsymbol{v}$ can be written as the gradient of some scalar: $\boldsymbol{v} = \boldsymbol{\nabla}\Phi$. A flow for which the above condition is satisfied, is called irrotational flow, a term which will become clear below.

[4] This term could be translated by "swerve" or "clash". Although it appears only once in the whole text of the *De Rerum Natura*, it has become closely associated with the Lucretian doctrine, partly, I think, because of its intriguing nature. Serres [210] has taken the occurrence of the *clinamen* in the poem as evidence for Lucretius' role as founder of modern physics, as the first to recognize the difference between "laminar" and "turbulent" motions, and as the first who recognized the importance of vortical flow.

[5] For a review of Leonardo's work on fluid mechanics see e.g. [269].

[6] For a more extensive description of Descartes's theory see e.g. [47, Ch IV] and [4].

The vector $\boldsymbol{\nabla} \times \boldsymbol{v}$ will be replaced by the vector $\boldsymbol{w}$, i.e.

$$\boldsymbol{w} \equiv \boldsymbol{\nabla} \times \boldsymbol{v} . \tag{1.1}$$

Its components can thus be written as:

$$\begin{aligned} w_1 &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ w_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ w_3 &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{aligned}$$

where $u$, $v$, $w$ are the components of the velocity vector $\boldsymbol{v}$ and $x$, $y$, $z$ are the components of the spatial location vector $\boldsymbol{x}$, i.e. the vector which determines the position in space of a fluid particle in a fixed frame of reference.

Despite the still small number of mathematical tools available at that time, the study of irrotational flows appeared feasible since they were completely described by the velocity-potential $\Phi$. However, it soon became apparent that the theory of these so-called potential flows hardly provided any flows relevant to the real world. Today, we know that most fluid motions in nature and technology are rotational. Even in nearly irrotational flows, the relatively small amount of vorticity present may be of central importance in determining major flow characteristics.

The vector field determined by $\boldsymbol{w}$, which is equal to zero for irrotational flows as we have seen, has become known as the **vorticity** field [7]. Although the mathematical concept of vorticity cannot be found literally in 18th century works on fluid mechanics, these works undisputably contained the first notions of the importance of rotational flows, i.e. flows in which $\boldsymbol{w} \neq 0$ in some parts of the flow. In the writings by D'Alembert and Euler two of the most prolific writers on fluid mechanics during the 18th century, formulations of an important equation in vorticity theory can be found, which has been called the **D'Alembert-Euler vorticity equation** [268, §94]:

$$\frac{D\boldsymbol{w}}{Dt} = (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{w}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) . \tag{1.2}$$

The derivative $D/Dt$ is the so-called material derivative:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} , \tag{1.3}$$

which is the rate of change when a material particle is followed during its displacement.

An important change that took place at the turn of the 18th century was the institutionalisation of scientific research and specialisation of scientists, brought about by the rise of scientific institutions.

For France, the most important institution became the École Polytechnique in Paris, where Laplace (1749-1827) and Poisson (1781-1840) propagated the *physique mathématique* as the way of describing all sorts of physical phenomena. For them, the physical reductions, i.e. the

[7] For the introduction of the term "vorticity" itself we refer to [268, §29]. For convenience, we will use this term from this instance on, though from a historical viewpoint this is incorrect.

making of models to represent physical phenomena, were equally important as the mathematical deductions, and consequently required the same rigorous methods. For Laplace these were algebraic methods.

One example of the mathematical treatment of a physical phenomenon was the theory of heat which Fourier (1768-1830) brought into the area of rational mechanics, at the same time drawing attention to a distinction between a mathematical and a physical representation. Another example is Fresnel who constructed a physical model of light, thereby abandoning the corpuscular theory of light initiated by Newton and giving a stimulus to a new, mechanical, view of the ether. Like *actio in distans* the ether had been haunting scientists ever since Newton had proposed it as the medium for the action of gravity and light (hence its description as "luminiferous medium").

Cauchy (1789-1857), another *Polytechnicien*, was one of the first, together with Lagrange (1736-1813), to introduce symbols to stand for the vorticity components. However, their early work, which appeared after that of d'Alembert and Euler, was still purely formal and somewhat mystifying.

The kinematical significance of the vorticity vector did not begin to be recognized until around 1840 the Irish scientist MacCullagh and Cauchy himself proved that the components of the curl-operator (see (1.1)) satisfied the vectorial law of transformation [268, Ch III]. By that time Cauchy had also provided a complete and explicit description of the convection of vorticity. One of his results, the **Cauchy Vorticity Formula** [268, §94], is given by:

$$\boldsymbol{w} = (\boldsymbol{w}_0 \cdot \boldsymbol{\nabla}_X)\boldsymbol{x} \tag{1.4}$$

or:

$$w_i = (w_0)_j \frac{\partial x_i}{\partial X_j}$$

where the scalars $X_i$ are the components of the material location vector $\boldsymbol{X}$, i.e. the location of a fluid particle at time $t = 0$ which can be regarded as the labels of the particle. This expression, a general solution to the D'Alembert-Euler vorticity equation (1.2), has the following physical interpretation, as illustrated in fig. 1.1: a cube, initially of sides $X_1$, $X_2$, and $X_3$, is deformed in time; the vector from one corner to the opposite, which represents the local vorticity vector, is thus stretched and rotated. These two important aspects of vortex dynamics will be called **vortex deformation**.

Cauchy also reformulated a result which had already been present in Lagrange's works. This **Lagrange-Cauchy Theorem** [8] says that inviscid flows, i.e. flows in which viscosity plays no role, which are irrotational at a certain moment, have been so ever before, and will remain so for ever.

The physical meaning of the vorticity vector only became clear in a paper of 1847 when George Stokes, longtime Lucasian professor in Cambridge, discovered that at each point of a velocity field the vector $\boldsymbol{\nabla} \times \boldsymbol{v}$ may be regarded as twice the angular velocity of a small element of the continuum [222, §2]. The same paper has appeared to be a treasure of many other important contributions to fluid mechanics. One of his results was a fundamental theorem on the kinematics of continua, nowadays called the **Cauchy-Stokes Decomposition Theorem**: an arbitrary instantaneous state of motion may be resolved at each point into a uniform

[8] See [268, §104] for the controversies which have surrounded this theorem during the 19th century.

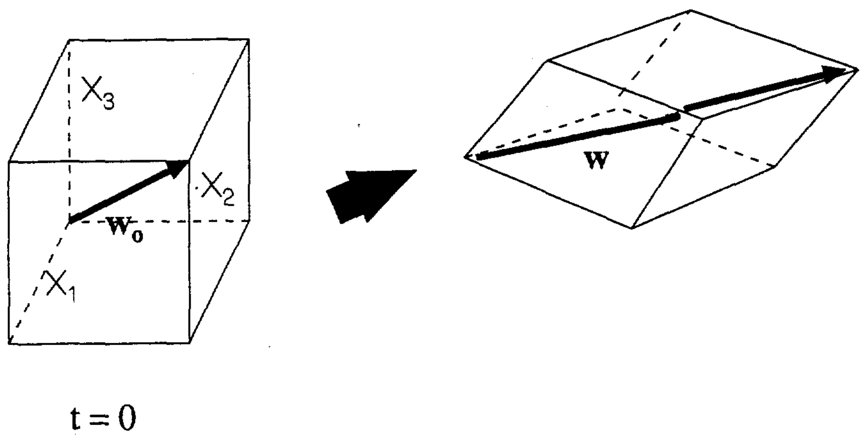


Figure 1.1: Illustration of vortex deformation as expressed by the Cauchy Vorticity Formula (1.4). The arrow indicates time development.

translation, a dilatation along three mutually perpendicular axes, and a rigid rotation of these axes [222, §2] [9].

This result would again be discovered ten years later by the German scientist Hermann von Helmholtz and led to the sound foundation of the theory of vorticity, as will be discussed in the next chapter.

[9]See also [268, §34]. A more general and abstract version of this theorem has become known as the Hodge Theorem in mathematics.

## Chapter 2

# Helmholtz's Contribution to Vorticity Theory (1858)

In Germany, towards the end of the 18th century and at the beginning of the 19th century, science had been strongly influenced by the romantic *Naturphilosophie* in which a speculative approach towards natural phenomena was advocated. All natural phenomena, both organic and inorganic, both the microcosmos and the macrocosmos, had to be united into one model. This conviction stimulated the use of analogies and interest in electricity and magnetism, not without success. It also led to a rejection of mechanistic explanations and of the existence of atoms. Newton's mechanistic approach of nature had to be superseded by a dynamical view of the world which was considered as one living whole.

In their search for knowledge, several German scientists were most of all guided by their intuition. The deductive method and the use of experimental data were almost completely absent. This strongly frustrated the German scientist Helmholtz, whose attitude, though certainly influenced by the Naturphilosophical doctrines, was different [1]. He was one of the first to treat all the phenomena which had seemed so different from each other in the 18th century (heat, light, electricity, and magnetism) as different manifestations of a new concept: energy. In 1847, this resulted in his mathematical formulation of the principle of conservation of energy.

Around 1857, Helmholtz, who had been trained as a physician and had become professor of physiology in 1849 in Konigsberg, was working on physiological topics and became involved in related areas like optics and acoustics. His study of the physiology of the ear incited his study of the application of Green's integrals to hydrodynamics, leading to a paper titled "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen", written in 1857 and published in 1858 [75].

Though, as discussed in Chapter 1, others before him had become aware of the fundamental importance of vorticity in fluid flows, Helmholtz can be given the honour of being the first to construct a rather complete set of theorems and equations describing the kinematics and dynamics of vortex motion. His achievement, an impressive demonstration of mathematical skill and physical insight, has become a classic and up to this day has been cited frequently and respectfully. Below, some important results due to Helmholtz are presented.

After some introductory remarks and citing Lagrange and Euler as predecesors, Helmholtz came to an analysis of the general movement of a small fluid particle (in old German: *Wassertheilchen*). He noticed that part of this movement is described by the vector $\boldsymbol{\omega}$, given by:

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{v} \tag{2.1}$$

which could be regarded as the rotation of the particle [2]. Comparing with definition (1.1), we

[1] For a full treatment of Helmholtz's scientific achievements, see [110].

[2] Compare the Cauchy-Stokes Decomposition Theorem mentioned at the end of §1.2.

see that twice this rotation vector is the vorticity vector $\boldsymbol{w}$, as Stokes had already noticed (see §1.2).

Furthermore, Helmholtz proposed some definitions of vortex structures still used today:

**vortex line**: *A curve which at each point in the fluid is tangent to the local vorticity vector* $\boldsymbol{w}$.

**vortex tube** (see fig.2.1): *The surface formed by vortex lines passing through some closed contour is called a vortex tube.*

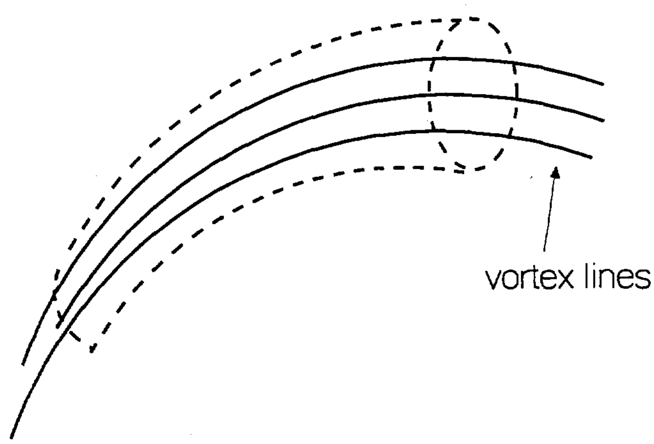


Figure 2.1: A vortex tube.

**vortex filament**: *A vortex tube (usually with an infinitesimally small cross-section) surrounded by irrotational fluid.*

In the paper one can find three important vorticity theorems which are still called after Helmholtz. We should remark that he only regarded perfect (inviscid) fluids and did not take into account the diffusion of vorticity. However, the first vorticity theorem is general and also valid for real, i.e. viscous, fluids. In order to get a clear understanding of this theorem, we have to introduce the following definition:

**strength of a vortex tube**: *The strength of a vortex tube at a certain cross-section A is defined as the surface integral*

$$\int_A \boldsymbol{w} \cdot \boldsymbol{n} \,.$$

We can now formulate

**Helmholtz's First Theorem**: *The strength of a vortex tube at a single time is the same at all cross-sections.*

From the First Theorem, Helmholtz correctly concluded that vortex tubes cannot end inside a fluid but must be closed or end at a boundary [3].

The other two theorems [4] are only valid for inviscid flows.
In order to state the second vorticity theorem we have to introduce the concept of

**material lines**: *A line in a vector field is material if it constantly consists of the same material particles.*

We can now formulate:

**Helmholtz's Second Theorem**: *Vortex lines are material lines.*

This means that, for the conditions mentioned above, vorticity can neither be generated nor destroyed. Vorticity is a property attached to the fluid particles and is transported by them.
Finally, we can state

**Helmholtz's Third Theorem**: *The strength of a vortex tube remains constant as the tube moves with the fluid.*

In §1.2, we saw that in the works of D'Alembert and Euler a vorticity equation can be found as given by (1.2). Helmholtz rediscovered this equation, though for incompressible flows only and his result is still called the **Helmholtz (vorticity) equation** [5]:

$$\frac{D\boldsymbol{w}}{Dt} = (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} \ . \tag{2.2}$$

This equation describes the vortex deformation phenomenon, already illustrated in fig.1.1 since the Cauchy vorticity formula is a solution of the Helmholtz equation [6].

Another important discovery by Helmholtz was the analogy between parts of the (older) electromagnetic theory and vorticity theory. Helmholtz's equation from which the velocity field $\boldsymbol{v}$ can be calculated, once the vorticity field $\boldsymbol{w}$ is given, is usually called the **rule of Biot-Savart** after its electro-magnetic counterpart.
The equation reads:

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\nabla} \times \frac{1}{4\pi} \int_{V'} \frac{\boldsymbol{w}(\boldsymbol{x}')}{r} = \frac{1}{4\pi} \int_{V'} \frac{\boldsymbol{w}(\boldsymbol{x}') \times \boldsymbol{r}}{r^3} \tag{2.3}$$

where $V'$ is the vorticity-containing volume and $\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x}'$; see fig.2.2.

In the last section of his paper, Helmholtz treated circular vortex filaments or infinitesimally thin vortex rings; see fig.2.3.

Showing his impressive mathematical skill, Helmholtz derived expressions for the velocity field induced by one vortex ring on another one and extended his result to an arbitrary set

[3]This conclusion is not true of vortex lines, as has sometimes been claimed. See [268, §10] for a discussion and for the proof that vector lines of any solenoidal field cannot possess any special properties.

[4]Helmholtz's own proof of the Second and Third Theorem are not completely rigorous [268, §46].

[5]Lamb, in his famous text-book *Hydrodynamics* (see §5.1), has pointed out a flaw in Helmholtz's derivation, which may, however, be corrected [115, §146].

[6]Helmholtz does not seem to have been aware of this fact.

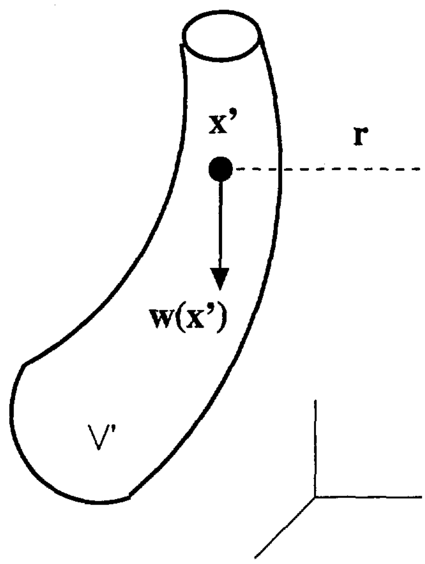


Figure 2.2: Illustration of the rule of Biot-Savart: the velocity at location $\mathbf{x}$ is determined by the vorticity $\mathbf{w}$ in vorticity-containing volume $V'$.

of coaxial [7] rings. From this he derived results for a single ring of infinitesimally small core radius and in infinite space. The ring's radius and velocity appeared to remain constant. At its center the fluid had a constant velocity along the ring's axis in the direction in which the ring moved, i.e. fluid flowed through the ring's aperture.

Helmholtz predicted that two coaxial rings, moving in the same direction behind each other, would show what has since been called the leap-frog effect: the ring at the back will approach the ring at the front; meanwhile the latter's radius is increasing and its velocity consequently is decreasing; at a certain moment the lagging ring will overtake the leading ring and the initial situation will emerge again though with rings exchanged. This procedure, Helmholtz thought, will repeat itself indefinitely.

Another situation discussed was that of two coaxial rings approaching each other (i.e. having opposite direction of vorticity). They would grow in size and approach each other at decreasing speed. Helmholtz remarked that this situation, if completely symmetrical, could also be obtained by letting a single ring approach a fixed wall perpendicularly.

After publication of this paper, Helmholtz again directed his broad mind towards acoustics and optics, in which areas he published several fundamental publications. Though occasionally he still worked on hydrodynamical topics, this research mostly grew out of his other fields of research [232, p.529]. Besides, he may have realized that further mathematical elaboration of e.g. the interaction of vortex rings would be very difficult, as would be discovered by those elaborating the vortex atom theory, introduced by Kelvin ten years after Helmholtz's seminal paper.

[7]Coaxial vortex rings: parallel rings having a common center axis line.

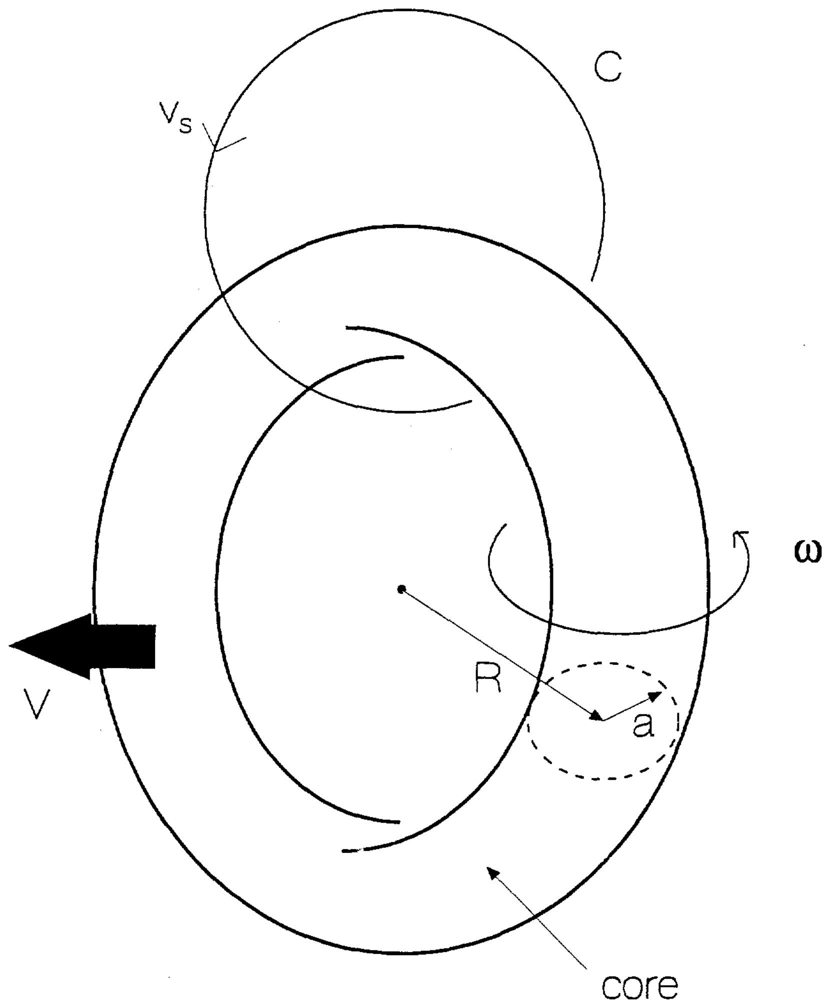


Figure 2.3: A vortex ring: $R$ = ring radius, $a$ = core radius, $\omega$ = angular velocity, $V$ = ring velocity. Curve $C$ is related to the definition of circulation (see §4.1).

Chapter 3

# Kelvin and the Road towards the Vortex Atom

Physics in the 19th century can be characterized by unification and mathematization. In the 18th century "unification" had only meant the interpretation of phenomena other than mechanical as so-called imponderable (i.e. electric, magnetic, and caloric) fluids. In the 19th century, the unification became much broader: an explanation of nonmechanical phenomena was tried by means of the methods which had been developed in mechanics. We have seen that mechanical concepts had already played an important role in Descartes' approach (see §1.2). In the last century, this approach revived and the formulation of mechanical (or mechanistic) models [1] became a popular activity, especially in Britain. The new "mechanical view of nature" saw matter in motion as the basis and explanation of all physical phenomena. It made possible a mathematical formulation of physics and freed it from ad hoc interpretations such as the imponderable fluids had been.

In Britain important developments started to take place at the beginning of the 19th century. The pursuit of scientific research became a serious profession and universities expanded their science faculties. In Cambridge, not only analytical mathematics was introduced from the Continent, also Cauchy's work on ether became known. The quest for an ether theory (see §1.2) still occupied the minds of many scientists in the 19th century. Stokes, as one of the first English scientists, began to elaborate the mechanical theory of the so-called elastic solid ether. In order to avoid any speculation about the molecular structure of the ether, Stokes stressed its physical structure. The same physical interest had become clear in his important paper of 1845 (mentioned in §1.2) in which not only theory of viscous flows but also the theory of "elastic solids" had been firmly established [222].

Stokes was a representative of a fairly large group of scientists, that became responsible for the revitalization of British science: the Scottish and the Irish. Whereas in the South of England (e.g. Cambridge) industry was despised and physics was mostly left to amateurs, in the North a rather opposite attitude could be found. The Scottish were much more nonconformists who turned to trade and, especially, technology. Other famous men of science from the North, who would become representatives of the new Cambridge school of physics, were William Thomson (1824-1907) (later Lord Kelvin) [2], Peter Guthrie Tait (1837-1901), James Clerk Maxwell (1831-1879), and Joseph John Thomson (1856-1940). All these men have played an important role in the story of the vortex atom.

Helmholtz's 1858 paper, a fine example of "applied mathematics", was received with great sympathy by British physicists. Among Helmholtz's greatest admirers was Kelvin, who had become professor of Natural Philosophy at Glasgow University in 1846 and would keep this

[1] Discussion on the meaning of the term "model" will be postponed to the Epilogue.

[2] In this thesis only the name Kelvin will be used in order to avoid confusion with another English scientist who will also be playing an important role in the development of the vortex atom, J.J. Thomson. One should realize that this is not really correct, as by the time W. Thomson wrote on the vortex atom, he hadn't been raised to the peerage yet.

position for 53 years. In 1855 both men, who had deep respect for each other, met for the first time, when Kelvin invited Helmholtz to attend a meeting of the British Association [110, I,p.252]. Kelvin's biographer, Thompson [232], holds that Kelvin had read Helmholtz's 1858 paper already in its year of publication. However, according to the recent, very extensive, biography by Smith & Wise [218], it was only in 1862, that Kelvin did become aware of Helmholtz's paper, when Tait drew his attention to it. Anyhow, in 1863 Kelvin told Helmholtz about his views on the power of vortices to explain the rigidity of matter, during a visit Helmholtz brought to Glasgow.

To get a better understanding of the birth of the vortex atom, we have to regard the road leading towards Kelvin's paper "On vortex atoms" of 1867. Apart from the influence of Helmholtz's 1858 paper, several aspects related to Kelvin's scientific development have to be taken into account, which will be treated in §3.1. The most direct incentive towards the vortex atom, Tait's 1867 experiment with vortex rings, is treated separately in §3.2.

## 3.1 Kelvin's Scientific Development towards the Vortex Atom

Up to 1847 Kelvin and others had succeeded in translating the observed analogies between the theories of heat, electricity, magnetic forces, and hydrodynamics into a single mathematical form, though each area still constituted a separate physical interpretation of the basic mathematical form. There was unity of language rather than of phenomena. The new dynamical approach of heat brought about an alteration of this state: the "mechanical effect" (energy) lost by gross bodies had to re-emerge in mechanical states of the ether and of constitutive elements which, for convenience, were referred to as molecules.

At the end of the 1840s, his work on the foundation of thermodynamics would also incite Kelvin to speculations on the molecular structure of matter. For the convertibility of heat and work he stated the hypothesis that heat is a form of energy, consisting of molecular motions. In this way, he violated one of the fundamental doctrines of his prior work: the opposition against physical hypotheses. The Scottish anti-hypothetical tradition had always limited natural philosophy to the sensible motions of bodies, without regarding molecular motions. All the same, this shift to the molecular level determined his further work.

During the 1840s, Kelvin also sought an explanation for the dynamics of force distributions. He stated important new concepts (e.g. energy), and transformed his kinematics of fields into field dynamics. One important source of inspiration in this regard was the work by Faraday (1791-1867). In his description of the forces between magnetic bodies, Faraday had introduced in 1845 the term "magnetic field", a concept that became of fundamental importance in the further development of 19th century physics. Faraday regarded the field as an intervening ether, but his representations couldn't explain the mechanism by which forces were propagated. According to Faraday, the transmission of forces takes place along the so-called "lines of force".

Kelvin rejected Faraday's view that all phenomena could be explained as "force" and also the view of the German romantic *Naturphilosophen* (mentioned in Chapter 2) that matter was a state of dynamic equilibrium between opposing "forces". In both views all attempts at mechanical reduction, such as Kelvin would like to formulate, had been despised. However, Faraday's results did stimulate Kelvin in attempting to find other, *in casu* mechanical, theories of the ether, which *could* explain the propagation of forces. Furthermore, around 1850 he came to a mathematical theory of magnetism in which Faraday's concept of lines of force was the central issue. Faraday's experiments on the influence of magnetism on polarized light, also stimulated Kelvin's first thoughts on "vortical" structures: he explained magneto-optic rotation as an elastic reaction in the ether to innate spiral structures that are also in orbital rotation.

During the years 1847-1851, Kelvin still maintained his purely macroscopic ideal of describing physical phenomena by a mathematical theory. However, this had become threatened by increasingly pressing demands for a physical conception of molecular reality. The mathematical analogy was powerful, but also showed weakness: related physical theories were set in parallel without relating them physically. To avoid this problem, Maxwell had proposed a less restrictive use of mathematical analogy. Kelvin could not agree with this approach and began to use the molecular theory and started his life-long pursuit for a theory of matter that would unify all physical forces.

Though not directly apparent, Kelvin's views on the ether must have influenced his view on matter, since he was convinced that there was no dichotomy between ether and matter. His opinion on the structure of ether would change fundamentally during these years. Initially, he had regarded ether as air, later he thought the ether to be much finer-grained than air. It was no longer air, it was like air [281, Ch.7]. The year 1851 showed an important transition for Kelvin in his constant search for a consistent theory of ether and matter [218, Ch.12]. He proposed the "aer" to indicate a unity of ether and matter. However, Kelvin's proposal to treat matter and ether as structures of the same kind in an underlying continuous fluid was regarded sceptically.

During this period, a typical example of a mechanical model was proposed by the Scottish physicist Rankine. In Rankine's theory of molecular vortices each atom of matter consisted of a nucleus surrounded by an elastic atmosphere. The quantity of heat was the kinetic energy of the revolutions or oscillations among the particles of the atmospheres, which Rankine supposed to constitute vortices about the nuclei [3].

With his theory, Rankine became one of the first to regard the mathematical consequences of the vortex hypothesis. Besides, he set the view on the ether as consisting of nuclei of atoms, vibrating independently (or nearly so) of their atmospheres. The model also impressed Kelvin, who in his paper introducing the vortex atom in 1867 (see §4.2 below) would remark that Rankine had showed the "possibility of founding a theory of elastic solids and liquids on the dynamics of ... closely-packed vortex atoms". To Kelvin this was "a most suggestive step in physical theory" [243, p.3].

Summarizing his views on the relation between matter and ether in a paper of 1856 [242], Kelvin mentioned three possible conceptions. Besides Rankine's notion of matter permeating the spaces between the ether's nuclei and the mechanical view in which matter and ether consisted of particles, he suggested an alternative model, showing his slow movement into the direction of the theory of vortex atoms. In the late 1850s he came to be convinced that the ether should be regarded as a fluid. The vortical motions in the perfectly elastic ethereal continuum were the cause of the molecular structure and the solidity and impenetrability of bodies. But an important question remained: how could his speculations provide a physical explanation if vortices in a plenum did not seem to possess the property of indestructibility?

Beside the requirement of indestructibility, another requirement showed up. The kinetic theory of gases, developed in the 1850s and 1860s, had encouraged the notion that vibrating molecules, supposed to consist of hard bodies, were the sources of spectral radiation. These motions were transmitted through the ether as vibrations of definite wavelengths. This view

[3] A fuller treatment of this theory, and its role in the development of thermodynamics, is given in [218, Ch.10].

had led to the requirement of flexibility and elasticity of the atom. Among the British physicists who tried to develop dynamical molecular models to explain results of spectroscopy, Kelvin would become very ambitious. Years after his first work in this field he would mention the fact that it was Stokes who, in 1852, had taught him the requirement that "the ultimate constitution of simple bodies should have one or more fundamental periods of vibration" [243, p.3].

Except elasticity and thermodynamics, electricity and magnetism started to play a role in Kelvin's formulation of a theory of molecular structures. In 1847 he had suggested to consider the propagation of electrical and magnetic forces in terms of the linear and rotational strain of an elastic solid. His starting point was Stokes' already mentioned 1847 paper [222] in which, for the first time in a clear manner, rotation and strain in continuous media had been treated mathematically. Kelvin regarded motions of electrical fluid as vortex motions and considered thermo-electric rotations [218, p.405-].

Stokes would also stimulate Kelvin's interest in hydrodynamics [4]. In 1857, Kelvin wrote Stokes on his attempts to find a theory of rotating "motes" in a perfect fluid. The stress which he had begun to put on rotational motion not only arised from his work in heat and magnetic theory, but he also thought that the repulsion caused by the rotating motes would lead to a stiff, stable structure required for e.g. luminiferous vibrations [218, p.409].

Kelvin's attempts to implement rotational motion led in 1858 to a very speculative thought on "eddies" in an universal fluid, which might explain gravity and inertia in the solar system. He had started work on the hydrodynamics of the motes, parts of the molecules, and their interactions and stability. In his correspondance with Stokes, we find the latter's criticial remarks, but Kelvin did not reply [218, p.411].

It seems that Kelvin would only start again on rotational motion after his acquaintance with Helmholtz's paper in the early 1860s [218, p.412], as will be discussed in Ch. 4.

## 3.2 Kelvin and Tait

Besides Kelvin's general convergence towards vortex motion as discussed above, a more direct incentive in the development of the vortex atom theory has to be mentioned. Kelvin's inspirator in this respect was Peter Tait, professor of natural philosophy in Edinburgh.

Like Kelvin, Tait admired Helmholtz and had made a personal translation of the 1858 paper directly after its publication [109, p. 127]. In a short epilogue to this translation, which appeared only in 1867 [75], Tait spoke of "one of the most important recent investigations in mathematical physics". We can only guess why this translation was published ten years after the original had been published, taking into account Tait's remark that he had made it "long ago". Tait mentioned Helmholtz's personal revision of the translation, though without an indication when this had taken place.

Both men became close cooperators in 1861 when Kelvin proposed Tait to join in writing a textbook [218, Ch. 11]. In 1866 and 1867 their collaboration was at its peak as they worked on what would become one of the most important and influential 19th century references in physics, the *Treatise on Natural Philosophy*. In the book they attempted to propagate the use of 'dynamics' rather than simply 'mechanics'. The significance of this choice in favour

[4] In a letter to Stokes of 1857, Kelvin wrote: "Now I think hydrodynamics is to be the root of all physical science, and is at present second to none in beauty of its mathematics." As Smith [217, p.400] correctly remarks, it would be absurd to regard these words as a "key to unlocking the mysteries of Thomson's inner thoughts"; they are a sign of his enthusiasm for this rising branch of physics.

of dynamical explanation was fundamental: instead of an abstract, purely analytical, mathematical treatment of motions, they chose a physical approach based on the assumption of Newton's laws of motion, highlighting the importance of the concept of force. Instead of only kinematical considerations, they put emphasize on the dynamical aspects. This program of dynamical theories implied replacing forces acting at a distance by matter in motion. All physical phenomena were dynamical, also those which appeared to be statical.

Because of the co-operation on the *Treatise on Natural Philosophy*, Kelvin visited Tait regularly. During one of those visits, in January 1867, Kelvin witnessed a simple experiment performed by Tait in his study, which can be regarded as the most direct incentive for the vortex atom model [5].

A description of Tait's experiment can be found in a letter Kelvin wrote to Helmholtz a few days after the experiment:

> Just now, however, *Wirbelbewegungen* have displaced everything else, since a few days ago Tait showed me in Edinburgh a magnificent way of producing them. Take one side (or the lid) off a box ... and cut a large hole in the opposite side [see sketch in fig.3.1]. Stop the open side AB loosely with a piece of cloth, and strike the middle of the cloth with your hand. If you leave anything smoking in the box, you will see a magnificent ring shot out by every blow. ... We sometimes can make one ring shoot through another, illustrating perfectly your description; when one ring passes near another, each is much disturbed and is seen to be in a state of violent vibration for a few seconds, till it settles again into its circular form. The accuracy of the circular form of the whole ring, and the fineness and roundness of the section, are beautifully seen. ... The vibrations make a beautiful subject for mathematical work. The solution for the longitudinal vibration of a straight vortex column comes out easily enough. The absolute permanence of the rotation, and the unchangeable relation you have proved between it and the portion of the fluid once acquiring such motion in a perfect fluid, shows that if there is a perfect fluid all through space, constituting the substance of all matter, a vortex-ring would be as permanent as the solid hard atoms assumed by Lucretius and his followers (and predecessors) to account for the permanent properties of bodies ... and the differences of their characters. Thus, if two vortex-rings were once created in a perfect fluid, passing through one another like links of a chain, they never could come into collision, or break one another, they would form an indestructible atom; every variety of combinations might exist. Thus a long chain of vortex-rings, or three rings, each running through each of the other, would give each very characteristic reactions upon other such kinetic atoms. I am, as yet, a good deal puzzled as to what two vortex-rings through one another would do (how each would move, and how its shape would be influenced by the other). By experiment I find that a single vortex-ring is immediately broken up and destroyed in air by enclosing it in a ring made by one's fingers and cutting it through. But a single finger held before it as it approaches very often does not cut it and break it up, but merely causes an

[5] Experiments on vortex rings had been few up to that time. Helmholtz, in his 1858 paper, had suggested a way to produce ring-like structure by means of a spoon, pulled through the surface of a water tank. However, it seems that he never undertook any serious experiments on vortex rings.

Early experimental work had been done by Rogers [197] in 1858 and by Reusch [187] in 1860.

indentation as it passes the obstacles, and a few vibrations after it is clear. [232, p.513]

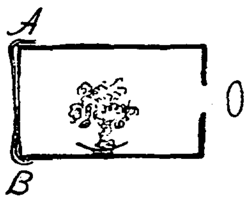


Figure 3.1: Kelvin's sketch of Tait's experiment. From [232].

From Kelvin's paper "On vortex atoms" [243] (to be discussed in §4.2) we learn that both men also experimented with two smoke boxes and studied the behaviour of two vortex rings approaching each other under different angles. Unfortunately, we only have a very concise, quantitative description of their observations [243, p.11-12]. In one of his popular lectures, first published in 1876 [229], Tait would perform the same experiments before an audience, where he also showed vibrations of rings obtained by using an elliptical or square hole in the box.

Clearly, the experiment with vortex rings caused the final convergence of Kelvin's views on vortex motion and only three weeks later, in February 1867, his theory of vortex atoms was exposed in a lecture before the Royal Society of Edinburgh.

## Chapter 4

# Kelvin and the Birth of the Vortex Atom

In the 17th century, the time of the Renaissance and revaluation of the classical Greek philosophy, the Democritean/Lucretian atom theory (see §1.2) had encountered a renewed interest. Not only physicists, but also chemists, looking for models able to explain chemical phenomena, came to favour a corpuscular-theoretical model [47, Ch III].

By the 1860s the Lucretian theory of matter was still generally accepted as a way to regard matter. At about the same time, the "hard bodies" had been taken up by the promotors of the kinetic theory of gases. One of them was Maxwell and the theory had quickly gained much support. It was a mechanistic-physical theory in which observable macroscopic properties of gases were deduced to the movements of molecules or atoms, hard particles (or bodies) which behaved according to Newton's laws. Its success strengthened the general belief in the reality of atoms [1].

Kelvin, however, did not favour the Lucretian theory of matter, and hence the kinetic gas model. The vortex ring, and the results in vorticity theory discovered by Helmholtz in 1858, not only led to his own theory of matter but also to some new fundamental results on vortex motion.

## 4.1 Kelvin's Contribution to Vorticity Theory

The year 1867 could be named Kelvin's *annus mirabilis* with respect to his work on vorticity, resulting in one short notice and two extensive papers: "On vortex atoms" [243] and "On vortex motion" [245]. These three publications can be read completely independently, though in the last we can detect how his vortex atom model [2] influenced the kind of research on vortex motion he thought necessary for the development of the theory.

Kelvin's short notice appeared as an appendix to Tait's translation of Helmholtz's 1858 paper [244] (see §3.2), having been sent as a letter to Tait shortly before the translation was published in the *Philosophical Magazine* [75]. It contained only one result, but this expression for the "translatory velocity of a circular vortex ring" has become one of the classical results in vorticity theory and it has since often been referred to and applied. It is given by [3]:

$$V = \frac{\omega a^2}{2R}(\log \frac{8R}{a} - \frac{1}{4}) \tag{4.1}$$

[1] See e.g. [31] for a general treatment of kinetic gas theories.

[2] The terms "vortex atom model" and "vortex atom theory" will be used without distinction. For a discussion of the vortex atom as a model, we refer to the Epilogue.

[3] This result is only valid for the vortex ring which we will call the Kelvin-ring. See §A.2 of the Interlude for its definition. In absence of further remarks, vortex rings in the discussion of 19th century research will mean rings with core size small compared to their radius and of uniform distribution of vorticity, as Kelvin assumed. Nowadays it is usual to add an order term to the expression for $V$. Kelvin realized the existence of lower order terms but could neglect them due to his assumptions.

where $V$ is the ring velocity, $\omega$ is the the approximate "angular velocity of the molecular rotation" in the core (see relation (2.1)), $a$ the radius of the ring's core, and $R$ the ring's radius; see fig.2.3.

Unfortunately, we get no indication about the way Kelvin derived his expression, but most probably he based it on the analysis of vortex rings which had appeared in Helmholtz's 1858 paper [4].

Though read in 1867, the paper "On vortex motion" would only be published in 1869, after it had been "recast and augmented" in 1868 and 1869. In Kelvin's notebooks of 1867-9 we find several calculations and drawings of vortices in preparation of this paper [218, p.422].

Contrary to the expectations raised by the title, the first part of the paper is devoted to "the hypothesis, that space is continuously occupied by an incompressible frictionless liquid acted on by no force, and the material phenomena of every kind depend solely on motions created in this liquid". Though it contains some references to vortex motion [5], we only mention his treatment of the topological concept of multiply continuous spaces, i.e. spaces of which the bounding surface is such that there are irreconcilable paths between any two points in it. A picture was presented of several knots showing a variety of knotted and knitted "wires"; see fig.4.1. Presumably, he needed this result to defend the existence of a large variety of vortex atoms (see §4.2).

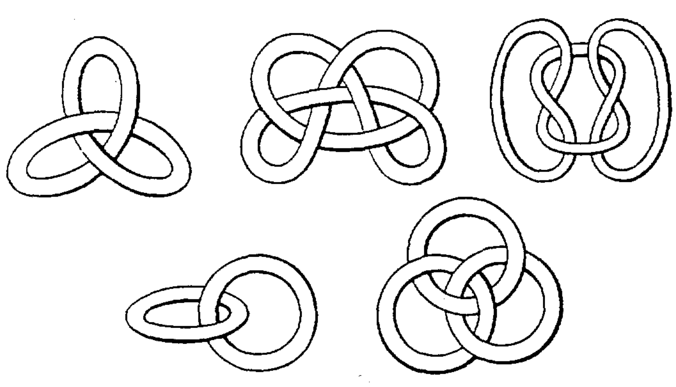

Figure 4.1: Knotted and knitted wires. From [245].

In the "instalment" of 1869, the second part of the paper, we find Kelvin's greatest contribution to vorticity theory, i.e. the introduction of the concept of circulation:

**Circulation:** *The circulation* $\Gamma$ *around a closed curve, say* $C$, *in some velocity field is defined in the following way:*

$$\Gamma \equiv \oint_C v_s \tag{4.2}$$

[4]Rott & Cantwell [201] remark that Helmholtz's derivation included an erroneous factor of 2, but this did not influence his final result as he used only a low order approximation. Kelvin derived the result for higher accuracy and apparently corrected Helmholtz's mistake.

See [212] for a recent discussion on Kelvin's probable source of inspiration.

[5]Kelvin's definition of a vortex [245, §20] seems to be the first attempt in history, but has become only one of many. A proper definition is still lacking.

where $v_s$ is the component of $\boldsymbol{v}$ tangential to the curve $C$ which is a loop enclosing any part of the ring's torus; see fig.2.3. We can regard $\Gamma$ as the mean value of the tangential velocity, multiplied by the length of the circuit [6].

It is Kelvin's merit that, due to the introduction of circulation, Helmholtz's Third Theorem and the Lagrange-Cauchy Theorem (see §1.2) could be proved elegantly and more rigorously.

Kelvin's reformulation of the Helmholtz's Third Theorem is now called

> **Kelvin's Circulation Theorem:** *An inviscid flow is circulation-preserving, i.e. the circulation in any closed line moving with the fluid remains constant through all time.*

Kelvin's reformulation of the Lagrange-Cauchy Theorem (see §1.2), i.e.:

> **Lagrange-Cauchy Theorem (Kelvin's version):** *A motion is irrotational if and only if the circulation about every circuit equals zero,*

confirmed his belief in determinism in the mechanical philosophy which he and Tait had proposed in their *Treatise on Natural Philosophy* (see §3.2). However, he also recognized that it was only valid for simply-connected regions in which all curves are reducible, i.e. can be shrinked indefinitely. This may explain his fascination for multiply-connected regions and the knotted and knitted wires mentioned above.

## 4.2 "On vortex atoms" (1867)

Kelvin's lecture "On Vortex Atoms" [7] [243] was delivered before the Royal Society of Edinburgh in February 1867, only three weeks after Tait's smoke ring experiment. Kelvin started with some severely critical remarks on the "Lucretian atom" [8]. To him justification for the "monstrous assumption of infinitely strong and infinitely rigid pieces of matter" [243, p.1] [9] could only be found in the fact that it allowed an explanation of the "unalterable distinguishing qualities of different kinds of matter" [243, p.1]. Furthermore, the Lucretian model didn't "explain any of the properties of matter without attributing them to the atom itself" [243, p.1]. Therefore, "the Lucretius atom has no prima facie advantage over the Helmholtz atom" [243, p.2], i.e. the vortex ring which Helmholtz had introduced in his 1858 paper.

Kelvin realized that a "gas" consisting of his vortex atoms would have to obey several requirements if it wanted to challenge the kinetic theory of gases. Referring to the experiment in Tait's lecture-room, he remarked that the elasticity of the rings was at least as good an explanation for the elasticity of gases as the "clash" of the Lucretian atoms [243, p.2], which to him could be equalled to the atoms of the kinetic gas theory. Contrary to investigators in this

[6] Having defined circulation, we can rewrite equation (4.1) in its commonly used form (assuming a uniform vorticity distribution in the core, as Kelvin had done):

$$V = \frac{\Gamma}{4\pi R}(\log\frac{8R}{a} - \frac{1}{4}) \tag{4.3}$$

where $\Gamma$ is the circulation of the ring.

[7] It will be referred to as the Vortex Atom paper in the rest of this thesis.

[8] The role of Lucretius in Victorian Britain is discussed in [270], where Kelvin is cited: "I have been reading Lucretius ... and trying hard on my own account to make something out of the clash [= clinamen] of atoms, but with little success" [270, p.348] (see §1.2).

[9] Wherever possible, for Kelvin's work page numbering of the *Mathematical and Physical Papers* (referred to as *MPP* in the Bibliography) will be used.

field, Kelvin remarked that for his theory of matter he only had to assume the rings' "inertia and incompressible occupation of space" [243, p.2].

He must have realized that if one wanted to investigate the properties of a gas of vortex atoms, their interactions would have to be calculated and that this would be very hard. Nevertheless, with some confidence Kelvin stated that "a full mathematical investigation of the mutual action between two vortex rings of any given magnitudes and velocities passing one another in any two lines, so directed that they never come nearer to one another than a large multiple of the diameter of either, is a perfectly solvable mathematical problem". And adding with complete confidence: "Its solution will become the foundation of the proposed new kinetic theory of gases" [243, p.2].

With regard to the requirement of the explanation of the variety of atoms, Kelvin showed diagrams and wire models "to illustrate knotted or knitted vortex atoms, the endless variety of which is infinitely more than sufficient to explain the varieties [...] of known simple bodies and their mutual affinities" (see §4.1 and fig.4.1). Helmholtz's First Theorem gave him confidence that the atom would be indestructible: "a closed line of vortex core is literally indivisible by any action resulting from vortex motion" [243, p.3].

With regard to the requirement of the capability of vibration and elasticity of the atoms and the related requirement on the explanation of spectral properties, Kelvin remarked that in case of the Lucretian model one had to assume that molecules cannot be single atoms, but are groups of atoms "with void space between them" [243, p.3]. However, "such a molecule could not be strong and durable", whereas vortex atoms had by nature "perfectly definite fundamental modes of vibration, depending solely on that motion the existence of which constitutes it" [243, p.4].

Kelvin admitted that elaboration of this last statement would be another difficult analytical problem, "but certainly far from insuperable in the present state of mathematical science" [243, p.4]. He pointed out that that this result could only apply to a vortex ring for which the core radius was much smaller than the ring radius. However, he tried to defend his model by remarking that the "lowest fundamental modes of the two kinds of transverse vibrations of a ring" were more serious than the deformation of the core [243, p.4].

To show the fruitfulness of his new model, he regarded sodium. Since the sodium atom had appeared to have two fundamental modes of vibration of slightly different periods, Kelvin concluded that the "sodium atom may not consist of a single vortex line, but might very probably consist of two approximately equal vortex rings passing through one another like two links of a chain" and this model would certainly "fulfil the 'spectrum test' for sodium" [243, p.5] [10]. Furthermore, Kelvin proposed to study the influence of temperature on the fundamental modes of the vortex atoms.

Clearly, Kelvin did realize that his new theory of matter had to satisfy several severe requirements: it had to compete with the kinetic theory of gases, it had to explain the variety and indestructibility of the atoms, and it had to show spectral properties. He also realized that to this end an analytical elaboration of the properties of both single rings and of the interaction of several rings was needed. However, to him the elaboration of these issues would only be a question of time and (great) mathematical effort.

[10] In a footnote, added afterwards, Kelvin remarked that the sodium atom might after all be explained by a single vortex ring.

# Chapter 5

# The Development of the Vortex Atom

In the introduction of Chapter 3 we mentioned the popularity of mechanical models among British scientists. We especially treated Rankine's molecular vortices, a typical example of these models. The vortex atom model, however, was a different kind of model. It didn't involve "mechanical" concepts like springs (as Kelvin would indeed frequently apply for his models), but was solely based on hydrodynamical concepts.

This "hydrodynamical" model meant a fundamental difference with the existing views on matter. This was clearly expressed by the English scientist Pearson [1] in 1889: "The old view endowed its atoms with certain inherent forces, and having done so, more or less completely ignored the existence of the medium; the new view endows its atoms with no inherent forces, but with motion - it looks to the action of this motion on the medium to explain the action of one atom on another. The old view saw everywhere in the universe force, the new view finds everywhere motion - that is a gross way of putting the difference" [174, p.71].

However, one soon started to realize, Kelvin himself not in the least, that many properties which had been attached to the vortex atoms in the Vortex Atom paper and other properties which might still be unknown, had to be fully elaborated before the value of the new theory of matter could be properly judged. Before discussing (in §5.3) the treatment of some of the problems related to the properties of the vortex atoms, both by Kelvin and others, we first look at the reception of the vortex atom in Britain (§5.1) and on the Continent (§5.2), *in casu* in Germany and France.

## 5.1 The Reception of the Vortex Atom in Britain

Initially, Kelvin found little interest for his theory of matter. For one part, this can be explained by the circumstance that he had opposed the Lucretian atom and thus challenged the foundations of the successful kinetic theory of gases. Kelvin's theory was based on continuous space-filling entities, a theory which could not find many adherents at that time. Another reason is the fact that his Vortex Atom paper remained the sole publication on this topic for several years after its publication. Only in December 1871, Kelvin spoke again to the Edinburgh Society on the continuation of his Vortex Motion paper. Unfortunately, only an abstract of this lecture has been published [247].

As for Tait, from whom one would be inclined to expect at least some support for Kelvin's theory, his first reaction was only published in 1875, anonymously in a book titled *The Unseen Universe* [221]. It was a typical example of the discussion on "natural theology" still prevalent at that time in Britain. Tait, together with his colleague Stewart, tried to defend the modern physics of their time against accusations of being too materialistic and hostile to religion. Tait's

[1] In 1885, Pearson would propose an atom model resembling the vortex atom model in a paper from which we cite here. It is exemplary for the research the vortex atom theory would set into motion in the 1870s and 1880s.

chapter on the vortex atom theory was mainly used for treatment of mind-matter dualism and did not contribute to a serious discussion on the vortex atom, since it was neither in favour of the new theory nor against it.

Though Tait considered the vortex atom theory a subject worthy for one of his "Lectures on some recent advances in physical science", first published in 1876 and still in print in 1885, his final words in his lecture on vortex motion sound very reserved:

> With a little further development the theory may perhaps be said to have passed its first trials at all events, and, being admitted as a possibility, left to time and the mathematicians to settle whether, really, it will account for everything already experimentally found. If it does so, and if it, in addition, enables us to predict other phenomena, which, in their turn, shall be found to be experimentally verified, it will have secured all the possible claim on our belief that any physical theory can ever have. [229, p.304-5]

The first real, and strongly needed, support for Kelvin's theory from the physical community was provided by Maxwell. Though around 1867 Maxwell was still an ardent supporter of the kinetic gas theory (see the introduction to Chapter 4), he actually had preceded the vortex atom model in 1861 by introducing an ether model involving so-called molecular vortices (see e.g. [66, pp.146-]). Initially, he had reacted rather critically to the new theory of matter in a letter to Tait of November 1867 [109, p.106]. However, in his address to the British Association for the Advancement of Science [2] in 1870 [145], he appeared to have changed his sceptical attitude, though his treatment of the vortex atom was mainly intended to show how illustrative methods or expositions could help to represent physical phenomena.

He recognized that when a full mathematical elaboration of the model could be achieved, the vortex atom would "stand in a very different scientific position from those theories of molecular action which are formed by investing the molecule with an arbitrary system of central forces invented expressly to account for the observed phenomena" [145, p.223]. These theories he referred to were the then prevalent atom models which had earlier been proposed by Boscovich and Dalton. Boscovich had proposed his point-atom model in 1745. He suggested that matter consisted of points without extensiveness which could be regarded as centers of force. Interaction took place through "fields" of force around the centers. During the 19th century, Boscovich's point-atomism was still in favour, although in modified versions, especially with the French physicists, and with certain philosophically minded German physicists. In Britain opinions were divided. It was strongly criticized for its artificial character by, among others, Kelvin [3]. The theory of matter proposed by the English physicist and chemist John Dalton (1766-1844) had been intended to link the theory of atoms to Lavoisier's theory of the elements. Though most chemists in the 19th century were sceptical about the existence of atoms, they recognized that this concept could be helpful in acquiring new insights in chemical phenomena [4].

Maxwell also realized that this model could not "fail to disturb the commonly received opinion that a molecule, in order to be permanent, must be a very hard body" [145, p.224], i.e. the opinion of those adhering the kinetic gas theory. According to Maxwell's molecular theory

[2] To be called the British Association furtheron.

[3] A short description of Boscovich's theory can be found in [65].

[4] This situation is well illustrated by a lecture by the English chemist Williamson [280] and the sequent discussion [29] from 1869.

of spectra, spectral lines were due to collisions of these hard bodies in diatomic molecules. However, experiments showed that mercury vapor consisted of monatomic molecules and *did* emit a spectrum. Maxwell realized that the vortex atom was a better model with regard to this issue which is reflected in his most important supportive contribution to the vortex atom theory, the "Atom" article in the 1874 edition of the Encyclopaedia Britannica [147]. There, he explicitly asserted that neither the Lucretian nor the Boscovichian atoms could account for spectra and that the vortex atom alone was capable of internal motions or vibrations.

As the greatest advantage of the vortex atom, Maxwell mentioned the fact that no "hypothetical forces" are introduced to "save appearances" [147, p.471]. He remarked that a theory of matter had to explain mass and gravitation, but apparently thought that the model would comply with this requirement. However, only three years later, Maxwell's expectations had faded away. In a review of a text-book on the kinetic theory of gases in *Nature* [144], he expected that the vortex atoms "would soon convert all their energy of agitation into internal energy, and the specific heat of a substance composed of them would be infinite" [144, p.245].

By 1877 the vortex atom (and vortex motion in general) had drawn general attention and Maxwell's criticism does not seem to have influenced opinions. Several highly esteemed mathematicians and physicists [5] started research in the area. Beginning the same year two Cambridge mathematical journals, the *Quarterly Journal of Pure and Applied Mathematics* and the *The Messenger of Mathematics*, began to be filled regularly with papers by undergraduates and younger fellows on vortex motion and the interactions of variously shaped bodies moving in perfect fluids. However, the papers were mostly fragmentary and contained little explanation on their motivation or their purposes. Besides, the Helmholtz's theory of vorticity still didn't seem to have been grasped by all scientists. An amazing lack of proper understanding of the theory can be found, for example, in a reaction on Kelvin's theory published in 1883 by the British geologist Croll [43](1883). Croll wondered about the force that counterbalances "the centrifugal force of the rotating material of the vortex-atom" and argued that it cannot be the exterior incompressible liquid surrounding the vortex atom, since it offers no resistance to the motion of the vortex atom. There is no cohesive force, Croll thought, so "What then prevents the revolving material from being dissipated by the centrifugal force of rotation?". Apparently, he didn't realize that vorticity was a property of the flow field itself.

The vortex atom even became an issue outside the physical community. In literature, the vortex atom was alluded to in George Eliot's novel *Middlemarch* [35]. S.T. Preston, private scholar and prolific writer in physics, tried to introduce the vortex atom into a lively discussion on the physical basis of the "phenomena of thought" which took place in Nature around 1880. According to him, these "phenomena" were influenced by "changes or permutations of which the molecules of matter were capable". The old "hard" (Democritean) atoms could only move, but the vortex atom had more kinds of permutations; its number of permutations was even infinite, as thought was itself [183].

The theory of vorticity and vortex rings also became a common part of the text-books on hydrodynamics which started to appear in the second half of the 19th century. The one which would become famous for more than half a century, and is even referred to today, was *Hydrodynamics* by Horace Lamb [6]. Originally, it had appeared under a different title already in 1879 [115], originating from the lectures Lamb had started somewhat earlier when still a

[5] Love published an important review of "English researches in vortex motion" in 1887 [133].

[6] See [91] for a history of this text-book.

fellow at Cambridge. That edition already contained a chapter on vortex motion (treating e.g. the leap-frog of two vortex rings), but the subject of vortex atom was not treated. In the 1895 edition [115], Lamb remarked that "the method of experimental illustration by means of smoke-rings is too well-known to need description here", referring to Tait's lectures [229]. On the vortex atom Lamb remarked that "this lies outside our province, but it has given rise to a great number of interesting investigations ..." [115, §166] but he recognized that the impulse to some of the investigations on vortex motion mentioned in his book were suggested by Kelvin's vortex atom theory.

The changing attitude of the Cambridge community towards hydrodynamics was stimulated once again when in 1873 Maxwell presided over the reformation of the famous Tripos examination at Cambridge and managed to have hydrodynamics included. The following year, Kelvin and Tait acted as examiners and introduced questions on motion of perfect flows and Helmholtz's theory of vortex motion.

Maxwell had only expressed his intuitive opinion on the vortex atom, but what was really needed was a mathematical elaboration, followed by a physical interpretation, of the properties of vortex rings, both of single ones and of several interacting ones. Only in this way the physical community could be convinced of the consistency and usefulness of the model.

Around 1880, Joseph John Thomson had come to Cambridge on scholarship, where he was completely dependent on University support for maintaining himself as a scientific worker. Therefore, he sought a branch of physics which could earn him some esteem and a stable position [239].

In 1881 the topic for the prestigious Adams Prize, to be awarded in 1883, was announced: "A general investigation of the action upon each other of two closed vortices in a perfect incompressible fluid. In particular it is suggested that the case of two linked vortices should be fully discussed, with the view of determining (1) whether any steady motion is possible, and (2) whether any motion can occur in which there are periodical changes in the forms and dimensions of the vortices." Kelvin probably had some influence in the choice of this topic as it was a logical problem arising from his Vortex Atom paper.

Thomson, who had already published two papers on vortex motion, submitted an essay entitled *A treatise on the motion of vortex rings* [234] [7] and won. It was the first comprehensive attempt to get an analytical picture of interacting vortex rings, after the first attempts by Helmholtz who, however, had only treated coaxial rings (see Chapter 2).

Thomson's faith in the vortex atom theory as a theory of matter seems to have been weak. In his Introduction, he had to remark that the vortex atom theory "cannot be said to explain what matter is, since it postulates the existence of a fluid possessing inertia", and his claim that it was "evidently of a very much more fundamental character than any theory hitherto started" [234, §1] sounds as a hollow phrase, expressed out of politeness. To him, the most important aspect of the new theory seems to have been the fact that the vortex atom theory allowed investigation into the mechanisms of intermolecular forces. Thus, it enabled one to form "much the clearest representation of what goes on when one atom influences another" [234, §1]. Not surprisingly, Thomson chose to restrict his activities to an elaboration of a gas consisting of vortex atoms, interacting with each other.

Therefore, he first needed a strong fundament with regard to the behaviour and interaction of vortex rings. In the first part of the *Treatise*, Thomson derived, from the equations which

[7] To be referred to as the *Treatise* in this thesis.

had been found by Helmholtz, quantities like momentum and kinetic energy for rotational flows. Furthermore, he calculated the velocity and stability of vortex rings. The topic of the second part was the interaction of two rings and one ring in the neighbourhood of a solid, i.c. a flat plate and a sphere. However, to enable mathematical elaboration he had to assume that the rings did not approach closer than several times their radii. The third part was on linked or knotted rings, whose interest had been roused by Kelvin's Vortex Motion paper (see §4.1). As Thomson assumed that the shortest distance was small compared to ring's radius, the tubes could be regarded as straight cylindrical vortex columns, which again facilitated the elaboration enormously.

In the final part, exposing his vortex theory of gases, Thomson had to admit that the theory was much to complicated to be treated in general, but he mentioned he had tried to derive results which could show some of the properties of a gas consisting of vortex atoms, such as chemical affinity. This part, not surprisingly, was most speculative and after publication met with several critical remarks. It induced Thomson to investigate several other aspects of gases, mainly published in the *Philosophical Magazine*, of whose board Kelvin became a member in 1871. However, in a paper shortly published after the *Treatise* [235] on electric discharge in gases, he clearly recognized that his gas model was not suited at all for his purposes. By then, his faith had largely faded away.

Thomson's *Treatise* did not remain unnoticed, which was partly due to the prestige of the Adams Prize. However, though his work was generally prized for its impressive mathematical achievements, criticism arose soon after its publication. In a review which appeared in *Nature* in 1883, Osborne Reynolds, professor of engineering in Manchester [8], found one inconsistency in Thomson's derivations and one, apparently fatal, flaw in the vortex atom theory itself [190].

In the final part of his *Treatise*, Thomson had attempted a derivation of Boyle's law. For that purpose, he had assumed that the velocities at the solid boundaries were small. Reynolds remarked that these boundaries also existed of vortex atoms and that no reason existed to make this supposition. The flaw was concerned with Thomson's proposal of a test experiment on so-called thermal effusion. According to the current theories of gas at that time, the pressures on the two sides of a diaphragm which were of unequal temperature had to be proportional to the square root of absolute temperature. From his vortex gas theory, Thomson had derived that the pressure will be proportional to the temperature raised to a power greater than one. This experiment would be crucial for the theory, but he realized that it was hard to perform. Reynolds remarked that he had already performed such an experiment around 1879 and that Thomson had probably meant the phenomenon of "thermal transpiration", since effusion, he remarked, was only a theoretical idea.

Reynolds proposed a more suitable experiment, which he supposed to be crucial. According to his view, the velocity of sound must be limited by the mean velocity of the vortex atom. As Thomson had shown that this mean velocity diminished with temperature, and as experimentally it had been found that the velocity of sound increased as the square root of temperature, Reynolds concluded "that the verdict must be against the vortex atom theory. However the vortex atoms are very slippery things, and we should like to hear Mr. Thomson's opinion before adopting one of our own" [190, p.195]. However, Thomson never seems to have reacted to Reynolds's criticism, but, regarding the reputation of *Nature* and of Reynolds himself, it must have been a severe blow to the status of the *Treatise*.

[8] Reynolds's fascination with vortex motion is evident from a lecture given in 1877 [189], in which he treated some experiments he had performed with smoke rings around 1876 [188].

Despite this seemingly fatal attack, Reynolds, and presumably many with him, didn't dare to really challenge a theory introduced by someone of such high esteem as Kelvin. Or they didn't think it necessary. Instead of criticizing the vortex atom, however, there was the possibility of attempting to adapt the original vortex atom in order to let it meet certain requirements. The most important of these attempts was made by Hicks, one of the scientists whom the 1881 Adams Prize had stimulated to take up research in vortex motion.

Hicks had found two properties of matter which were hard to explain by the vortex atom theory: gravity and the different densities of the elements. The first, he thought, could be solved by considering the theory of pulsating spheres in a fluid, a phenomenon which had been initiated by the Norwegian physicist and founder of meteorology Bjerknes and which Hicks had already considered in a paper of 1879 [78]. He had found that gravitation could be explained if the circulations of the vortex atom exceeded a certain amount; this "cyclic irrotational motion, connected with the vortex, may be so large as to produce [a vacuum]" and hence he suggested to consider hollow vortices, i.e. vortex tubes whose vorticity was concentrated on their surfaces.

Regarding the different densities of the elements, Hicks had to invoke the ether. In an abstract [80] to the first of a series of three extensive papers on vortex motion ([81], [82], and [87]), he remarked:

> When the exceedingly small density of the ether compared with what we call ordinary matter is considered, it is clear that the supposition that matter is composed of vortices of the same density as the ether is surrounded with great difficulties, and we are driven to the conclusion that, if a vortex ring theory be the true one, the cores of the vortices must be formed of a denser material than the surrounding ether, and that probably this core has rotational motion. [80, p.305]

For Hicks it was evident that for an explanation of the different masses, the original vortex atom theory could not remain as elegant as it had been proposed by Kelvin. In the first two of three papers mentioned above, written in 1884-85, Hicks fully treated the hollow vortex rings and studied their steady motion and vibrations. Under his assumptions, he claimed, problems with Kelvin's vortex atom could be solved to any order of approximation.

Hicks's introduction of the hollow ring must have impressed the physical community for its ingenuity, but whether it could save the vortex atom theory remained unclear. However, Hicks himself soon left any attempts to extend his hollow-vortex theory, though in 1885 he received the Hopkins Prize, given for the best original discovery by an alumnus of Cambridge in mathematical physics in the previous three years. His faith to the vortex atom remained. In his address to the British Association of 1895 [85], he tried to explain static electricity by means of vortex atoms and in a lecture read in 1898 on "a kind of gyrostatic aggregate", which "has brought to light an entirely new system of spiral vortices" [86, p.332], he still wondered whether his new theory could throw any light on a vortex atom theory. However, at the end on this lecture he remarked to have found no point in pursuing his new results as he realized that it was "wild speculation" and that attention would be low.

We have to conclude that though outwardly the vortex atom seemed to be received with sympathy, a closer look reveals that severe and fundamental criticism appeared. However, even worse was the lack of appropriate attempts to defend the new theory of matter by elaborating its characteristics. Besides, it is difficult to judge whether the esteem for the vortex atom was due to the (supposed) scientific value of the theory, or to the status Kelvin had acquired, or just to the attractive experiments with rings.

## 5.2 The Reception of the Vortex Atom outside Britain

For reasons mentioned before, Kelvin's vortex atom was most heartily received in Great Britain itself, at least in the first instance. Reception elsewhere, which seems to have started slowly, largely differed among countries and among scientists. The treatment in this section of some of these reactions is especially meant to show distinctive nationalistic tendencies in physics.

Generally, on the Continent reception was hostile or, to say the least, indifferent. In both Germany and France philosophical objections hindered scientists, especially those who could be regarded as most able to do so, to contribute to the vortex atom theory. However, since no really convincing *physical* arguments were raised against the theory, Kelvin and his followers must not have been really disturbed by the reactions of their foreign colleagues (supposed they knew them).

Our survey of reactions surely is incomplete and will be restricted to Germany and France [9]. As for Russia, for example, we have evidence that the theory was known in some circles, but inaccessibility of Russian literature hinders a sketch of its reception there. Reception in the United States seems to have been passive only.

### 5.2.1 Germany

As remarked in Chapter 2, in the beginning of the 19th century German science had been largely influenced by the romantic *Naturphilosophie*, characterized by a search for underlying unitary principles and conservation laws and by an absence of the, typically British, empirical approach. However, by the time of Kelvin's introduction of the vortex atom, the romantic *Naturphilosophie* had largely lost its influence and several German physicists had developed a, sometimes extreme, desire for empirical evidence.

One of them was Helmholtz. After his 1858 paper, Helmholtz had returned to his research in acoustics and optics. In 1870, he wrote Kelvin that he was still working on vorticity theory, though only occasionally [232, p.529] and apparently not on the the vortex ring. Soon after his 1858 paper, he became more and more convinced that agreement between theory and experiment could only be acquired if viscosity was taken into account [111, p.23] and he realized that this would be fatal to the vortex atom theory.

Published comments by Helmholtz on Kelvin's vortex atom are scarce. In a funeral oration of 1871, he remarked:

> Ueber die Atome in der theoretischen Physik sagt Sir W. Thomson sehr bezeichnend, dass ihre Annahme keine Eigenschaft der Körper erklären kann, die man nicht vorher den Atomen selbst beigelegt hat [a remark from the Vortex Atom paper; see §4.2]. Ich will mich, indem ich diesem Ausspruch beipflichte, hiermit keineswegs gegen die Existenz der Atome erklären, sondern nur gegen das Streben aus rein hypothetischen Annahmen über Atombau der Naturkörper die Grundlagen der theoretischen Physik herzuleiten. ... Man hat begriffen dass auch die mathematische Physik eine reine Erfahrungswissenschaft ist; dass sie keine andere Principien zu befolgen hat, als die experimentelle Physik. [76, vol.III,p.13]

The only direct reference to the vortex atoms can be found in Helmholtz's preface to Hertz's famous book *Prinzipien der Mechanik* of 1894 in which we find a more fundamental reason for Helmholtz's passive attitude towards, Kelvin's theory:

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[9]For a view on the reception of the vortex atom in The Netherlands, we refer to papers by the physicists Lorentz [132], W.H. Julius [97], and V.A. Julius [96], and especially to the thesis by Quint [184].

> Englische Physiker, wie Lord Kelvin in seiner Theorie der Wirbelatome ... haben sich offenbar durch ähnliche Erklärungen [i.e. deriving all known physical laws from certain fundamental principles, e.g. from Newton's laws] besser befriedigt gefühlt, als durch die blosse allgemeinste Darstellung der Thatsachen und ihrer Gesetze, wie sie durch die Systeme der Differentialgleichungen der Physik gegeben wird. Ich muss gestehen, dass ich selbst bisher an dieser letzteren Art der Darstellung festgehalten, und mich dadurch am besten gesichert fühlte; doch möchte ich gegen den Weg, den so hervorragende Physiker, ... eingeschlagen haben, keine prinzipiellen Einwendungen erheben. [77, p.xxi]

Hertz's approach, as Helmholtz remarked, had been similar to Kelvin's. Not surprisingly, then, Hertz regarded Kelvin's theory as a firm support for his own hypothetical approach:

> Ich erinnere ... an die Wirbeltheorie der Atome von Lord Kelvin, welche uns ein Bild des materiellen Weltganzen vorfuhrt, wie es mit den Prinzipien unserer Mechanik in vollem Einklange ist. [77, vol.III,p.44]

Somewhat surprisingly, the vortex atom got a relatively important place in Oskar Meyer's well-known and influential text-book on the "rivalling" kinetic gas theory [155], whose first edition was published in 1877. For Meyer the vortex atom was "die glücklichste Hypothese" which could satisfy the requirements of a theory of matter. However, Meyer's arguments for his enthousiasm seem doubtful and highly speculative. Regarding the chemical aspects, he admitted that he couldn't mention many results which had been derived from the vortex atom theory. Apparently, however, the vortex atom was more than a *Hypothese* to him as he supposed that the ringlike form of the vortex atom could represent the "abgeplatte oder auch langgestreckte Form" of most molecules. Even in the 1899 edition of his book, by which time the vortex atom had become almost completely obsolete, a section on the vortex atom was inserted in which Meyer suggested that electricity might be included in the model.

With the decline of the Naturphilosophie, new schools of thought on the development of science arose in Germany. Lasswitz, a Gymnasium teacher of mathematics and physics and prolific essayist on epistemology, was a follower of the so-called neo-Kantian school whose principles are evident in his discussion of the vortex atom theory in a paper of 1879 [120]. Though initially Lasswitz praised the vortex atom theory and thought that eventually one could even explain "die Gesetze der Wärme und die Thatsachen der Chemie aus der Energie und der Form der Wirbelatome", he also thought it was still missing an essential element.

For him, a theory could only be of "wissenschaftliche Bedeutung" if it was "nicht bloss in irgend einem Theile der Physik von praktischem Vortheil", but also satisfied "das Erkenntnissbedürfniss des Geistes". "Der Bau der Wissenschaften muss ein einheitlicher sein." To him, a theory had to be extended "bis eine einfache Anschaulichkeit gewonnen ist; sie muss uns nachweisen, wie durch das Zusammenwirken unserer Sinnen und unseres Denkens fundamentale Begriffe unseres physikalischen Erfahrung erzeugt werden, bei welchen der ganzen Natur unserer Organisation nach eine weitere Frage nach Erklärung nicht mehr auftreten kann" [120, p.279]. This required an investigation of the concept of matter in a manner which Lasswitz called "erkenntniss-theoretisch".

Lasswitz concluded that the vortex atom was not an acceptable model. Kelvin had been able to propose his theory because "er und die Vertheidiger seiner Theorie immer noch die Atome als real-transcendente Dinge ansehen, nicht als Erzeugnisse unserer Erkenntnissthätigkeit bei unserer Orientierung in der Welt." Kelvin's vortex atom moved problems from the macrostructure of nature to its microstructure since it was still a moving and changing object; therefore, Lasswitz argued, the hard (Lucretian) atoms were preferable.

Though his arguments must have appeared vague and irrelevant to most physicists, especially the British, the rising popularity of the neo-Kantian attitude may be an important explanation for the lack of enthousiasm, and interest, for Kelvin's theory in Germany.

### 5.2.2 France

The strong bond between experimental and mathematical research in French science (see §1.2) had led to a flourishing scientific community and a spreading influence in Europe at the beginning of the 19th century. However, around 1830 this influence had started to decline and Britain and Germany were taking its place. Part of this shift can be attributed to the rapid adaption of education to the changed scientific attitude in these latter countries.

Recognition of the importance of vorticity had began early in France thanks to the results found by Cauchy (see §1.2), who in 1827 had introduced the concept of "rotations moyennes" (see e.g. [125]). However, the admiration of this preceding work, together with an even more chauvinistic admiration of the Cartesian heritage, hindered the French in a full appraisal of foreign vortex theories [10]. This is exemplified by Wurtz's book on atomism of 1873 [286], in which the author remarked that the vortex atom idea was not new, but essentially Descartes's theory: "l'esprit humain semble tourner dans un cercle". However, Wurtz admitted that Kelvin had used more rational scientific arguments.

In an early French reaction of 1870, Bertrand, professor at the *École Polytechnique*, argued that the existence of vortex atoms was not consistent with the equations of fluid mechanics. However, his arguments in the *Journal des Savants* of which Bertrand himself was the editor, remain unclear to us [25] (see also [268, §29]). Apart from short references, and even some appraisals, of the vortex atom theory (see e.g. [125] and [63]), around 1890 French interest in Kelvin's work seems to have been completely absent. Poincaré had chosen the vortex theory for his 1891-1892 lectures at the Faculté des Sciences in Paris [180], and the published version of the lecture notes can be regarded as the first text-book on this theory, which he considered the greatest achievement of fluid mechanics at that time. However, he only shortly mentioned Kelvin's vortex-atom theory, and did not even refer to Kelvin in his introduction of the circulation concept.

The French attitude may be explained by the general rejection of the manner in which physics in Britain was exerci and which the French saw exemplified by Kelvin's work. One of the most important critics in this regard, together with e.g. Poincaré, was Pierre Duhem, a highly prolific scholar on the history and philosophy of science and a respected physicist. He reproached the British scientists lack of order, method, and concern for logic and experimental results. Furthermore, he criticized the provisional character of the various models they had introduced and the incompatibility of these models.

Taking notice of Duhem's strong opposition to atomistic theories, it is not surprising that he was especially critical of the vortex atom. An example of his sometimes furious, and not completely objective, treatment of Kelvin's model can be found in his *L'evolution de la mécanique*, where Duhem remarked that "cette hypothese de W. Thomson nous presente le plus haut degré de simplification auquel puisse parvenir l'explication des phénomènes naturels" [52, Ch XIV]. Though Duhem admitted that it contained advantages such as the absence of

[10] Even in the 20th century, in France highly speculative books could appear (like those by Parenty [172] and Varin [271]) in which Descartes's vortex theory forms the basis of broad physical theories.

"force reelle", and the possibility to explain diversity of the elements, he warned that the vortex atom hypothesis "s'enfonce si profondement au-dessous des apparences sensibles, qu'il devient bien malaise de remonter jusqu'à celles-ci et de fournir l'explication des faits que nous constatons chaque jour."

Duhem's opinions were broadly propagated by himself and his own scientific work testifies that indeed he had chosen for a completely different approach of physics. His views on the differences between British science and French (and German) science must have been upheld by many of his contemporaries and has been a main reason for the lack of interest in the vortex atom on the Continent.

## 5.3 Issues surrounding the Vortex Atom

In the Vortex Atom paper, Kelvin had tried to show that his theory of matter possessed several of the properties which at that time were generally imposed on such a theory:

- the indestructible and impenetrable nature of the vortex atoms meant satisfaction of the requirement of conservation of matter;
- their elasticity and vibrations could explain the spectra;
- their many possible configurations could provide all elements with a signature of their own;
- no need existed for an artificial mechanism to keep several atomic rings together in a molecule; this explained chemical affinity.

The lack of any arbitrary parameters to be fixed made the model even more attractive. Besides, it was recognized that the vortex atom had an external kinetic energy, due to its self-induced velocity, and an internal form of energy, its vibration. For Kelvin, the only problem seemed to be the mathematical elaboration.

However, Kelvin must soon have started to realize that several problems were not only of mathematical nature. In the 1871 continuation of his Vortex Atom paper [247], mentioned in §5.1, he gave the description of three topics on which he intended to make further investigations: a system of vortex atoms as a kinetic theory of gases "without the assumption of elastic atoms"; the "realisation by vortex atoms of Le Sage's 'gravific' fluid consisting of an innumerable multitude of 'ultramundane corpuscles'"; and the "propagation of waves along a row of vortex columns alternately positive and negative". Still highly optimistic, Kelvin concluded that "the difficulties of forming a complete theory of the elasticity of gases, liquids, and solids, with no other ultimate properties of matter than perfect fluidity and incompressibility are noticed, and shown to be, in all probability, only dependent on the weakness of mathematics" [247, p.576-7].

As evidenced by the papers on vortex motion which would follow the next years, we conclude that by 1871 Kelvin already had planned a rather complete, and ambitious, research program in order to strengthen the position of his theory. In the next sections we discuss some of these fundamental issues, Kelvin's contribution to these, and reactions by others. The issues can be divided into two groups. The proof of steadiness and stability was a general problem related to vortex motion. Problems more directly related the vortex atom model were the comparison with properties exhibited by kinetic gases and the explanation of gravity and spectra.

### 5.3.1 Stability and Steadiness

Fundamental to the viability of the vortex atom model was the stability and steadiness of the vortex rings. The first issue was related to the question: is a vortex configuration conserved under changes of its level of energy? For the second, the problem read: are all (stable) states of a vortex configuration similar at a constant level of energy?

For Kelvin, the question of stability was familiar. Before the introduction of the vortex atom, he and Stokes had already discussed the stability of vortical motes (see §3.1). During the years 1872-1876, they discussed the topic again but this time it was concerned with a two-dimensional vortex (a cross section of a straight vortex tube [11]) in a circular boundary [12]. Both men could not agree. Kelvin argued that it was stable or at least quasi-stable and according to Stokes it was instable [218, p.433-].

Kelvin's picture of the stability of vortex atoms was of the same kind as that of the 2-D vortex. To tackle the issue, he regarded the change of the vortex from a stable maximum energy state to stable minimum energy state. The intermediate states, corresponding to different stages of vortex atom during its interactions with other vortex atoms, were called "maximum-minimim" states. To Kelvin these latter states meant that stability was uncertain, while Stokes argued that they would not be stable at all, except under special conditions.

Kelvin was also convinced that any finite number of vortices would always equilibrate, as it would gradually dissipate its energy and reach a state of minimum energy, i.e. state of maximal stability. The vortex atoms would then still be real vortices and energy in the universe would still be conserved. Stokes, who had clearly lost faith in the vortex atom model, tried to proof that the rotational motion of vortex atom could be annihilated, but to Kelvin nothing that God created could be destroyed [218, p.437].

In the Vortex Atom paper, Kelvin had tacitly assumed that "Helmholtz's rings" were stable and steady. Only in 1875, he seriously regarded these issues in a paper entitled "Vortex statics" [248]. His aim was formulated as "to investigate general conditions for the fulfilment of this proviso [i.e. steady motion], and to investigate, further, the conditions of stability of distributions of vortex motion satisfying the conditions of steadiness" [248, p.115]. His "general analytical condition for steadiness of vortex motion" had already been developed in the discussion with Stokes, as discussed above:

> If, with ... vorticity and "impulse" given, the kinetic energy is a maximum or a minimum, it is obvious that the motion is not only steady, but stable. If, with same conditions, the energy is a maximum-minimum, the motion is clearly steady, but it may be either instable or stable. [248, §4]

Though Kelvin realized that the energy of the vortex ring was a maximum-minimum, he presented it as a case of stable steady motion. For more complicated vortex structures, such as the "toroidal helix" [13], he could only speculate on their steadiness and stability.

[11] Kelvin's interest in this vortex may be due to the vortex which had arised in the work based on a centrifugal pump designed by his brother James [218, p.412-].

[12] In general, Stokes's reaction to Kelvin's ideas had become unfavourable. On Kelvin's theories of ether-matter interactions he remarked: "It is easy to frame plausible hypotheses which would account for the results, but it is quite another matter to establish a theory which will admit of, and which will sustain, cross-questioning in such a variety of ways that we become convinced of its truth" [282, p.xxxix].

[13] The nomenclature was mostly invented by Kelvin himself. For a description of this "toroidal helix" we refer to Kelvin's paper.

Kelvin realized that his results were rather unconvincing. "Hitherto, I have not indeed succeeded in rigorously demonstrating the stability of the Helmholtz ring in any case" [248, §19]. From a simple and purely intuitional consideration of the ring, he concluded that "from the maximum-minimum problem we cannot derive proof of stability" [248, §19]. For rings with "ordinary proportions of diameters of core to diameter of aperture", he could only rely on "natural history", i.e. the observation of his own smoke rings.

Kelvin also realized that as important as the stability of the single vortex ring would be the stability (and steadiness) of configurations consisting of several vortices. One of the questions which he wanted to solve in this respect was the greatest number of columnar vortices or rectilinear vortex tubes that could be put in a "vortex mouse-mill", a regular polygon with vortices at the corners. He supposed this to be a proper simplification of the much too complicated case of interacting vortex rings.

In 1878, he discussed results of an experiment by the American Mayer [249], who had put floating bar magnets vertically in a basin of water and had found their positions of stable equilibrium to be at the vertices of regular polygons with one in the middle. Since infinitely thin straight vortex columns interacted in the same way as the magnets, this experiment could solve Kelvin's question, which to him was "of vital importance in the theory of vortex atoms" [249, p.140]. However, though Kelvin could partly confirm Mayer's results by mathematical calculation, he didn't conclude on the stability of his atoms [14].

The number of investigations of the stability of vortex motion by others seem to have been small. In his *Treatise*, Thomson remarked that Kelvin (possibly referring to [248]) had "proved that [the vortex ring] is stable for all small alternations in the shape of its transverse section". Thomson himself concluded "that it is stable for all small displacements" [234, §13]. Investigating the steadiness and stability of two linked vortices, he found that this configuration was steady only when the rings were close together. Hicks, in his extensive papers on vortex motion [81] and [82], investigated steadiness and stability of his hollow vortex ring. He found stability for the ordinary hollow ring, but could only find (conditional) stability if he extended his ring model with additional circulation and density differences (see Chapter 4). In his final discussion of the vortex atom in 1895 [85], he still warned that this issue had not been solved.

The issue of stability and steadiness remained unsettled, mainly due to a lack of appropriate mathematical techniques. However, for Kelvin and his followers, the existence of smoke rings seemed enough "evidence", though they may have felt somewhat uneasy with the situation.

### 5.3.2 Compatibility with Kinetic Gas Theory

As mentioned in the introduction of Chapter 4, the popularity of the kinetic gas theory had largely increased by the time Kelvin had introduced his vortex atom. Kelvin himself strongly rejected the kinetic gas theory of elastic-solid molecules colliding by actual contact since he supposed that all kinetic energy would be converted into vibrations and rotations. However,

[14] See [219] for a full account of Mayer's experiments and also for J.J. Thomson's reference to Mayer's results in his speculation on a possible arrangement of electrons in atoms.

In his *Treatise*, Thomson treated the case of the "mouse-mill" analytically and showed that the motion was stable for the number of vortices $n < 7$ and instable for $n \geq 7$ [234, §54]. Only in 1931, Havelock showed that the case $n = 7$ is neutrally stable. Re-examination 50 years afterwards by Dritschel [49] (numerically for finite-core-sized vortices) and by Dhanak [46] (analytically) showed that $n = 7$ is stable; Dritschel pointed out the mistakes in Thomson's calculations.

Kelvin realized that to show the superiority of his own vortex atom model, he had to show that it possesed the same characteristics as kinetic gases. This would require a complete determination of the interactions of large numbers of vortex atoms, and a statistical approach was necessary as was common in kinetic gas theories. However, he had always despised such an approach as it introduced indefiniteness.

His only paper in this regard concerned one of the main results of the kinetic gas theory, i.e. the partitioning of energy: any concentration of energy within a gas had to spread throughout the whole gas, giving a specific randomized distribution. In "On the average pressure due to impulse of vortex-rings on a solid" of 1881 [252], he argued, without any proofs, that the pressure exerted by a cloud of vortex atoms (the "gas" which had been regarded by Thomson; see §5.1) was the same as that shown by the kinetic gas theory.

However, Kelvin also correctly realized that the integral of the pressure on the wall would be zero and thus the "gas" was not able to transmit momentum as any kinetic gas could do. Unconvincingly, he tried to show that the integral was nonzero if the flat plate was replaced by a finite tube [15].

Pressure was also regarded in Thomson's *Treatise* which would remain *the* attempt to settle the theory of a vortex atom gas. In his attempt to derive Boyle's law, Thomson found an additional term from which he concluded that "the vortex atom theory explains the deviation of gases from Boyle's law", adding the remark that other models were not able to show this result [234, §56].

However, Thomson realized that his derivation in the *Treatise* had been somewhat obscure. Besides, he might have been incited by Reynolds's fundamental criticism (see §5.1). In a paper of 1885 [237], he tried to apply a statistical method to derive again an expression for the pressure that a system of vortex atom exerts in a vessel. First of all he warned that the problem of the distribution of velocities of ordinary solid particles as founded by Maxwell and Boltzmann (the Maxwell-Boltzmann equipartition theorem) had been based on identical particles, whereas for the vortex atom model the sizes could differ. Nevertheless, after a long investigation on the distribution of vortex atoms, he indeed derived Boyle's law. However, this time one quantity remained undetermined, for which he couldn't indicate how to determine it.

The investigation of the relation between temperature and velocity of the rings, which had already been proposed for investigation in Kelvin's Vortex Atom paper [243, p.11], led to Thomson's discovery of a remarkable property of the vortex atom. For increasing temperature, i.e. kinetic energy, of a vortex ring its radius increased and consequently, according to expression (4.3) [16], its velocity decreased. For a kinetic gas, however, it had been shown that the velocity of the particles increased. Thomson tried to weaken this last result by remarking that it was based on monatomic gases; for diatomic gases, he thought, the velocity could indeed decrease [234, §57].

Despite Thomson's attempt, this property of the vortex atom would become much criticized. The only one who seems to have tried to remove the opposition on this issue was Hicks. However, his arguments only appeared in 1895 [85]), by which time the vortex atom theory had largely been abandoned.

Meanwhile, Kelvin realized that the only way to avoid more criticism was to intensify his attack on the kinetic gas theory. In a lecture before the British Association in 1884 [254],

[15] This result indeed seems to be in error, as has been already recognized by Quint in his thesis of 1888 [184] (see note 9) and recently by Saffman [205, §5.2].

[16] For which Thomson had, erroneously, found a factor 1 instead of $\frac{1}{4}$ [234, §13].

he tried to convince his audience that by his vortex atom model he had shown that inelastic bodies could behave as elastic bodies by their motion. He accused the kinetic theory of gases of being of no use on the atomic or molecular level and, surprisingly, claimed that the problem of equipartition of energy was deadly for any kinetic theory like the kinetic gas theory. Besides, he repeated his old argument that kinetic models had to assume elasticity, whereas the vortex atom model had not.

Unfortunately, reactions by the defenders of the kinetic gas theory are unknown, but we may suppose that the attempts by Kelvin and his followers were not taken very seriously, as their argumentation had been weak and unconvincing.

### 5.3.3 Gravity and Inertia

Ever since Newton's days, gravity had attracted the attention of scientists and, not surprisingly, many theories had been formulated [17]. The question how to include gravity in his vortex atom model must have bothered Kelvin soon after the appearance of the Vortex Atom paper. In 1868, he had started a correspondence with Fleeming Jenkin [18] who had immediately raised several questions regarding the properties of the vortex atom model. One of these concerned gravity, an issue Kelvin had not mentioned in his Vortex Atom theory.

A related issue, also raised by Jenkin [217], was inertia. In the Vortex Atom paper Kelvin had remarked that the only properties required of the vortex atoms were "inertia and incompressible occupation of space" [243, p.2]. Jenkin asked Kelvin how the fluid, possessing inertia, "can leave a free passage to aggregate vortices called gross matter." He could not understand "how the inertia of the medium in a given space can be increased or diminished by motion". Kelvin seems to have replied by explaining the difference between "what he saw as apparent inertia and primeval inertia, the latter only being inherent" but, unfortunately, details are lacking.

Also in Maxwell's contribution to the Encyclopaedia Britannica [147] (see §5.1) the need for an explanation of mass had been recognized. To Maxwell it seemed that Kelvin had proposed that only by the motion of the rings we can define matter in the primitive fluid, i.e. matter as a mode of motion. However, Maxwell remarked, the inertias of this mode of motion had to be explained, because "inertia is a property of matter, not of modes of motion" [147, p.472]

Thus, Kelvin could not avoid an attempt to comprehend gravity and inertia within the vortex atom theory if it were ever to achieve completeness. He tried to formulate such an explanation by means of the 18th century theory of gravity proposed by Le Sage, which had been introduced to him by Jenkin during their discussion on gravity. Le Sage's theory of gravitational action essentially depends on the bombardment of so-called ultra-mundane corpuscles on ponderable bodies [19].

In "On the ultramundane corpuscles of Le Sage" of 1872 [246], Kelvin extensively discussed Le Sage's theory of matter and added his own adaption to the vortex atom model. He noticed that the postulate of hard atoms in a void underlying both the kinetic theory of gases (which he had much criticized, as we have seen) and Le Sage's theory, was open to doubt and he tried to show that the specific problems he pointed out for Le Sage's theory could be resolved by replacing the hard atoms by his vortex atoms. The weakness of Kelvin's attempt to incorporate Le Sage's model was soon recognized. Nevertheless, Maxwell, in his enthousiastic

[17]See [137] for an extensive survey of conceptions of gravity in the 18th and 19th centuries.

[18]Jenkin, an engineer who had collaborated with Kelvin in submarine telegraphy, had been the first to treat the vortex atom theory after its introduction, in a paper on the atomic theory of Lucretius [94].

[19]For a full account of Le Sage's theory, see [137, p.111-].

Encyclopaedia Britannica paper [147], didn't dare to criticize the vortex atom model on this point despite his description of a fundamental flaw in Le Sage's theory.

Kelvin himself, recognized the weak position of his theory and explained its failure to provide support to his vortex atom in 1881 [253]:

> Le Sage's theory might give an explanation of gravity and its relation *to inertia of masses*, on the vortex theory, were it not for the essential aeolotropy [= non-isotropy] of crystals, and the seemingly perfect isotropy of gravity. No finger-post pointing towards a way that can possibly lead to a surmounting of this difficulty, or a turning of its flank, has been discovered, or imagined as discoverable. Belief that no other theory of matter is possible is the only ground for anticipating that there is in store for the world another beautiful book to be called *Elasticity, a Mode of Motion.* [253, p.473]

An alternative model to explain gravity came from Hicks, whose research on pulsating spheres has already been mentioned in §5.1. Hicks [78] tried to show how vortex atoms could show attractive and repulsive forces by a pulsative change of their volumes. However, he had to conclude that even for an incompressible fluid, for large collections of vortex atoms gravity "would take time for its full effect to travel any distance" [78, p.284]. This obviously meant an important flaw of his model and Hicks's proposal gained little support [137, p.285]. In a paper of 1883 [80], Hicks had become much more cautious. Though he could show that pulsating rings would also attract or repel each other, he made no remarks on gravity anymore.

### 5.3.4 Spectra

In §3.1. we mentioned that the origin and nature of line spectra of elements had been established by the middle of the 19th century and were found to originate in the motions of the molecules or atoms which were transmitted to the ether as vibrations of definite wavelengths. This led to the requirement of elasticity of the atom. Besides, different materials showed different spectra and any theory of matter should be able to show this. Kelvin realized that if he would indeed be able to show correct spectral properties of his atoms, this would mean an important step in the challenge of the kinetic gas theory, which, as remarked in §5.1, was insufficient on this issue.

In his Vortex Atom paper, Kelvin had tried to convince his audience, by means of his experiments with smoke rings, that "the vortex atom has perfectly definite fundamental modes of vibration, depending solely on that motion the existence of which constitutes it [i.e. vortex motion]" [243, p.4]. In "Vortex Statics" of 1875 [232] he derived the first four of these modes from an anlogy with the deformation of an elastic wire; see fig.5.1.

However, he realized that mathematical treatment would, again, be complicated and he proposed to consider the modes of vibration of an infinitely long, straight, cylindrical vortex. He added that "these results are, of course, applicable to the Helmholtz ring when the diameter of the approximately circular section is small in comparison with the diameter of the ring ... "[243, p.4]. Kelvin's results on the vibrational properties of this "columnar vortex" were only published in 1880 [250]. He did indeed find the dispersion relation for long bending waves on a rectilinear vortex with a constant-vorticity core, i.e. helical disturbances, but didn't dare to draw any further conclusions regarding the vortex atom model.

Though Kelvin's elaboration of the vibration of the column was recognized as a fine piece of analytical performance, many doubted the possibilities of the vortex atom model regarding

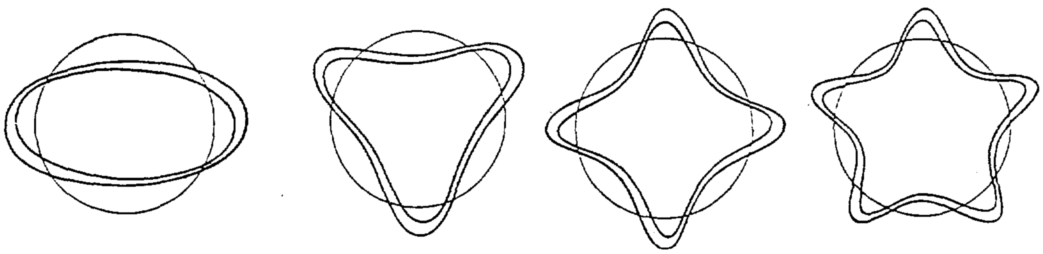

Figure 5.1: Steady modes of a vortex ring according to Kelvin. From [248].

spectra. In Maxwell's final deadly blow on the vortex atom theory in his critical 1877 paper [144] (see §5.1), he remarked that spectral requirements only allowed a finite number of degrees of freedom, whereas the vortex atom had an infinite number.

Hicks [80], on the other hand, seriously tried to elaborate the explanation of spectral lines by means of his hollow vortex model. However, his calculations did not show any concrete results and he found himself forced to remark that it would be "venturesome" to draw conclusions with respect to properties of the vortex atom, or to find analogies with kinetic theory of gases. One of the reasons for his reserve was related to the fact that, contrary to the interaction of hard bodies in kinetic gases, the interaction of vortex atoms depended on the mode of approach. Furthermore, his evaluation of the dependence of the time of vibration on energy showed disagreement with the result found by Thomson in his *Treatise.* This dependence of spectra on energy seemed fatal to the vortex atom.

Other, more detailed, critical remarks on Kelvin's attempt to derive spectral results appeared and were even more destructive. W.H. Julius (see note 9) remarked that a statistical approach was necessary for a real comparison of the properties of a gas of vortex atoms. One had to determine whether such a gas would give a single value of the spectrum if the number of "collisions" was reduced i.e. if the temperature was lowered. From a numerical example for carbonic acid, he showed that for a correct spectral property, the vortex atoms should not approach more than 30 times their diameter; evidently, there was no reason why this should be the case in a "gas" of vortex atoms [97, p.132-6]. Another Dutchman, Quint (see also note 9), remarked that Kelvin's suppositions in his treatment of the oscillations of the vortex column were in violation of the law of continuity [184, p.70]. Furthermore, he argued that Kelvin had just supposed that vortex atoms were elastic. His only proof seemed to lie in Tait's experiment, but Quint wondered whether this argument was not a hypothesis itself. The experiment had been carried out in a viscous, compressible fluid, so translation to ideal fluids was questionable [184, p.129].

The spectral properties of the atoms remained obscure and the results found showed severe inconsistencies. Not surprisingly, in the discovery of the spectral series formulae, by Balmer and Rydberg around 1885 (who were presumably only slightly familiar with Kelvin's theory), the vortex atom would play no role at all [151, p.172-]. Up to 1890 progress in the explanation of spectra was slow. By then the electromagnetic "viewpoint" was quickly gaining influence (see §6.3). Though still new models were proposed to reconcile older models like the vortex atom with the developing electromagnetic view [20], it had become clear that the search for explanation of spectra by models like that of the vortex atoms had been in vain.

[20] One of these was the model by Stoney in 1891, who suggested a connection between electrical charges associated with each bond in a chemical atom called "electrons" and vortex atoms [151, p.188-9].

Chapter 6

# The Decline of the Vortex Atom

In 1884 Kelvin was invited for a series of lectures in the United States at John Hopkins University, which have become known as the Baltimore Lectures [241] [1]. These lectures, devoted to the propagation of light and its interaction with matter, still showed a seemingly fully-resistent optimism regarding explanation by means of mechanical models: e.g. Kelvin presented a mechanical model of a molecule consisting of spherical shells connected by springs (called gyrostats). The vortex atom model, which we have called "hydrodynamical" to contrast it with the mechanical models (see the introduction of Chapter 4), was not even mentioned once [2]. In general, after 1884 the vortex atom would be absent in Kelvin's still steadily produced papers on all kinds of physical subjects.

Nevertheless, the vortex atom theory still played some role in physics, a role which Kelvin himself might never have expected and to which he didn't contribute very much himself. It formed the source of inspiration for a new direction in the modelling of the ether: vortex ethers. This is the subject of §6.1. As we have seen in Chapter 5, reception of the vortex atom had been unfavourable to Kelvin. Despite his efforts to elaborate the several issues mentioned in §5.3, and due to the flaws which these efforts had shown in the vortex atom model, the fame of his theory steadily declined. In §6.2, we will discuss Kelvin's reaction. Besides the insufficiency of the model itself, its decline can be related to the general conceptual changes which took place in physics in the 1890s. These changes, which made the fall of any theory like that of the vortex atom inevitable, will be treated in §6.3.

## 6.1 Vortex Ethers

The second (1904) edition of the Baltimore Lectures shows the fast developments which had been taking place at the end of the last century. It is extended with several appendices, containing some of Kelvin's lectures and papers which had appeared after the first edition. One of them was Kelvin's 1900 paper on two "nineteenth century clouds over the dynamical theory of heat and light" [259]. The first of these clouds concerned the relative motion of the still mystical interaction of ether and ponderable bodies which had become known as the ether drift: is the ether taken along with bodies (such as the earth) or is it always at rest; or: does ether drag exist or not?

Around 1885, the answer to this question was still lacking and the elaboration of ether-models had become a serious business. The elastic-solid ether model (see the introduction of Chapter 3) seemed useful indeed but also showed difficulties in connection with reflection and refraction. MacCullagh had proposed a rotationally elastic ether, but it could not be translated into a physical conception of its mechanism. Kelvin himself, in his Baltimore Lectures, had proposed a gyrostatic model which was based on MacCullagh's proposal.

[1]The first edition appeared the same year. A modern annoted edition is [99].

[2]Recall Duhem's criticism of Kelvin's easy shifting between various models (see §5.2.2).

Because of its characteristic of moving freely through the ether, the original vortex atom appeared to be a popular alternative in the 1880s to both Fresnel's ether drift with partial ether drag and Stokes's total ether drag. Since it was generally assumed that planetary bodies had to be able to move unresistedly through the ether, one realized that many of its properties had to be those of a perfect fluid. On the other hand, it was realized that in an ether model also properties of solids had to be included to make the ether capable of transmitting waves of light. Consequently, the question was raised whether a configuration of vortex atoms could indeed transmit any kind of "waves".

The relation of the vortex atom and the ether has a complicated history. Initially, in his Vortex Atom paper, Kelvin had not mentioned the ether at all. However, the problem must have started to bother him after one of the first letters he received from Jenkin (see §5.3) in which the question was put whether the ether also consisted of some kind of vortex rings. Unfortunately, we do not have Kelvin's response but we have found no indications that the perfect fluid in which the motion of the vortex atom took place was not the ether itself, as some historians have suggested (e.g. Wilson [281, Ch.7]). Both on account of his old (ether is aerial) and new (ether is air-like) vision on the substance of ether (see §3.1), it is probable that indeed Kelvin would think the ether to consist of vortex rings.

Some followers of Kelvin's theory of matter did identify the "perfect liquid" in which the vortex atoms existed with the ether, as is evidenced by their papers. In Hicks's contribution to the vortex atom theory, for example, a question had been raised concerning the explanation of the large density of ponderable matter as compared with ether (see §5.1).

If, on the other hand, the "perfect liquid" could not be equalled to the ether, the unpleasant situation of "a dualistic physical conception" arose, as was remarked by Pearson in his 1885 paper [174] (from which we quoted in the introduction to Chapter 5). If an atom is not a difference of motion in the ether, "we are compelled to suppose two primary substances, ether-substance and atom-substance". The problem is that "we should be explaining our atoms by means of an ether which would in itself be atomic" [174, p.119].

For Kelvin, Pearson's dualistic view seems to have been out of the question. He realized that one of the ways to settle the question would be to show the possibility of the transmission of waves by a "vortex ether". The only work which seems to have been intended for this purpose is [255], presented at the 1887 meeting of the British Association in Manchester. Though his lecture was titled "On the vortex-theory of the lumniferous ether (On the propagation of laminar motion through a turbulently moving inviscid liquid)" [256], the main title may just have been meant to draw the audience's attention, since in reality the paper is an attempt to "investigate turbulent motion of water between two fixed planes". The paper does not give the impression of a man believing to have proposed another promising ether model. On the contrary, it showed the signs of a man full of doubts, who at the end of the paper had to conclude with the "Scottish verdict of *not proven*" because he was still doubtful about the stability on the arrays of vortex rings (see fig.6.1) which he supposed to transmit the "waves of laminar motion".

Somewhat earlier, in 1885, another strong adherent to the vortex ether [3] had proposed his model. In 1880, FitzGerald, professor at Trinity College in Dublin, had already developed a

[3]The vortex ether has also become known as the vortex sponge. Though the introduction of this term has generally been attributed to Kelvin, he does not seem to have done so in the context of his ether model. We have only traced "vortex sponge" once, in a general paper on the stability of vortex motion [251].

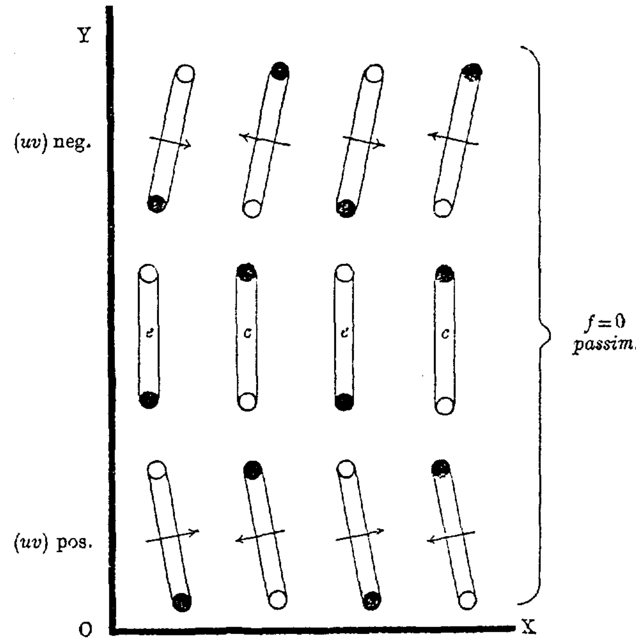


Figure 6.1: Kelvin's model of a vortex ether. From [256].

theory of the ethereal plenum to incorporate Maxwell's theory of light, based on pure electromagnetic properties. However, Kelvin's papers on vortex motion had changed his mind towards a theory in which ether and matter were represented by vortex motions in a universal plenum [4].

In "On a model illustrating properties of ether" [57] of the same year, FitzGerald proposed his new ether model. He stressed the point that regarding the nature of matter he would not make any supposition, the 'sponge' was just a model of the ether. In the rest of the paper he regarded a possible explanation of polarization, but drew no conclusions.

FitzGerald's paper does not seem to have roused much reaction. The only other paper related to this topic seems to have been a letter in *Nature* in 1889 [59] in which FitzGerald reacted to Kelvin's ether model, treated above [256]. His reaction consisted of a similar "electromagnetic interpretation of turbulent liquid motion" as in Kelvin's paper. To which he added that " a natural hypothesis would be that matter consisted of free vortex rings." [59, p.34].

Kelvin in his turn reacted to FitzGerald's remarks by regarding the "stability and small oscillation of a perfect liquid full of nearly straight coreless vortices" [257]. The most remarkable aspect of this paper is Kelvin's apparent conversion towards Hicks's hollow vortex core (see §5.1). The rotational vortex cores had to be discarded absolutely, and "we must have nothing but irrotational revolution and vacuous cores" [257, p.202]. He concluded that Hicks's work on the hollow vortex together with his own paper on the columnar vortex [250] (see §5.1) "will be the beginning of the Vortex Theory of ether and matter, if it is ever to be a theory" [257, p.202]. Apparently, his faith was declining.

[4]Despite his enthusiasm for the vortex ether, regarding the vortex atom, FitzGerald was very critical. In a paper of 1885 [56], he called the vortex atom "hardly ... an adequate theory ... It certainly is not sufficient to explain luminous and electrical and magnetic phenomena, to suppose the ether to be simply a perfect liquid at rest" [56, p.340].

After the introduction of a new vortex ether model, consisting of a regular pattern of straight hollow vortex tubes, Kelvin concluded:

> I have been anxiously considering the effect of free vortex rings with vacuous cores among the vortex columns of this tensile vortex ether, as suggested for cored vortices at the end of your communication ... [i.e. FitzGerald's paper [59]]. It will be an exceedingly interesting dynamical question; though it seems to promise at present but little towards explaining universal gravitation or any other property of matter; so you may imagine I do not see much hope for chemistry and electro-magnetism. [257, p.204]

These remarks seem to have ended the life of the vortex ether on both FitzGerald's and Kelvin's side, though in his address at the 1888 meeting of the British Association, the former still suggested that the problems could be overcome [58, p.562]. Ironically, nowadays FitzGerald is best known for the contraction theory named after him and Lorentz, which at the end of the 19th century was one of the signals that the ether would definitely disappear as a topic in physics.

In the meantime, a crucial experiment had been performed by Michelson and Morley in 1887 which showed the incorrectness of any stationary ether hypothesis and actually meant the death of the ether hypothesis. The properties of ether had suddenly become self-contradictory [5].

The Michelson-Morley experiment did not immediately put an end to model-building. On the contrary, Hertz's discovery of wireless waves, an experimental demonstration of Maxwell's electromagnetic theory, must have stimulated new research, as that by Hicks and Lodge.

In 1885, Hicks had discussed the possibility of transmitting waves through a medium consisting of an incompressible fluid of closely-packed small vortex rings [83] and in 1888 had proposed "a vortex analogue of static electricity" [84]. He still discussed the possibilities of a vortex 'sponge' model in his address to the Section of Mathematical and Physical Science of the British Association in 1895. However, Hicks had to conclude that "we can make little further progress until we know something of the arrangement of the small motion which confer the quasi-rigidity [of the ether]" [85, p.601], but he nevertheless considered some possibilities. He even tried to show how the explanation of "the magnetic rotation of the plane of polarisation of light" could be explained from vortex rings and how gravity could be explained from the vortex ether. "In all cases, whether a fluid ether is an actual fact or not, the results obtained will be of special interest as types of fluid motion" [85, p.606].

Still in 1893, Lodge [6], then professor of physics in Liverpool, argued that all phenomena and all experiments, except that by Michelson and Morley, could be explained in terms of the vortex atom. Probably, he suggested, even this last result could be "explained away", though

[5] Still, in a series of lectures given in 1899, Michelson [156] stated that "the 'ether vortex theory' which, if true, has the merit of introducing nothing new into the hypotheses already made, but only of specifying the particular form of motion required." After an explanation of his experiment with Morley, he treated the characteristics of vortex rings: "In fact, there are so many analogies that we are tempted to think that the vortex ring is in reality an enlarged image of the atom." Apparently, at that time Michelson still strongly believed in the unification which the ether could provide to physics though deep inside he must have had different opinions.

[6] In 1885 Lodge had done some really fundamental work to determine the value of the vortex atom [130]. To approach the difficult subject of interaction vortex rings, he had calculated and drawn their streamlines, a much more "experimental" approach than had been done in other papers before by anyone else. In the same paper he mentioned his experiments with smoke-rings using a "powerful intermittent induction-coil Leyden-jar discharge" [130, p.70].

on the other hand the vortex atom model might have to be adapted [131] [7].

Clearly, the adherents of the vortex ether were unable to put up a really satisfactory model. Their papers remained full of suggestions and lacked concrete results. Consequently, the role of vortex models never became generally accepted.

Not surprisingly, Kelvin turned his attention towards new ether models. His final opinion on the ether was published shortly before his death in 1907 and shows his transition towards the new "electric particle" approach (see §6.3 below). Looking back, he concluded:

> I do not propose to enter on any atomic theory of ether. It seems to me indeed most probable that in reality ether is structureless. ... We sometimes hear the "luminiferous ether" spoken of as a fluid. More than thirty [8] years ago I abandoned, for reasons which seem to me thoroughly cogent, the idea that ether is a fluid presenting appearances of elasticity due to motion, as in collisions between Helmholtz vortex rings. Abandoning this idea, we are driven to the conclusion that ether is an elastic solid. [262, p.236]

However, by 1907 the ether had already become an obsolete concept in physics due to the introduction of Einstein's theory of relativity.

## 6.2 Kelvin's Reaction to the Decline of the Vortex Atom

From the discussion in §5.3 of Kelvin's elaboration of the several issues surrounding his vortex atom theory, it appeared that his mind was still directed towards its problems at least up to his 1884 lecture before the British Association (see §5.3.2). Thereafter, as discussed in §6.1, Kelvin shortly directed his attention towards the vortex ether, though with an apparent lack of enthusiasm and hope.

In 1883 he still lectured on the vortex atom in Newcastle, where he received a copy of Thomson's *Treatise* [232, p.1046]. Not surprisingly, this support for his theory pleased him very much; to one of his colleagues he wrote: "I am becoming hot on vortex motion through having ... J.J. Thomson's book at hand" [213, p.212]. Though the same year, Tait wrote Kelvin that he had found a means for destroying the vortex atom for good, apparently Kelvin was not dissuaded by Tait's proof [213, p.214].

However, the still rising popularity of the vortex atom at that time and the attempts to apply the vortex atom theory to physical phenomena had also had its negative consequences: the weakness of the theory became clear. One of Kelvin's biggest worries must have been his inability to proof the stability of the vortex ring (see §5.3.1). However, in a footnote to a paper of 1905, we learn that Kelvin's doubts on this point had only become a real conviction after writing a paper entitled "On the stability of steady and of periodic fluid motion" published in 1887 [255]:

> It now seems to me certain that if any motion be given within a finite portion of an infinite incompressible liquid originally at rest, its fate is necessarily dissipation to infinite distances with infinitely small velocities everywhere; while the total kinetic energy remains constant. After many years of failure to prove that the original

[7] Actually, Lodge performed an experiment of the same rank as that of Michelson and Morley which, by its confusing result, aided to the declining faith in the ether [65].

[8] From this remark a problem of chronology arises, as by 1877 Kelvin was still favourably inclined towards the idea of elasticity resulting from vortex motion. He probably meant twenty years, as by 1887 his faith in the stability of the vortex ring had definitely been lost (see §5.3.1; see also [281, p.178]).

> Helmholtz circular ring is stable, I come to the conclusion that it is essentially unstable, and that its fate must be to become dissipated ... . [261, p.370]

Regarding the fate of the vortex atom theory with regard to other issues - gravity, spectra, and the compatibility with the kinetic gas theory - we only have some scattered remarks by Kelvin. In 1886, he told Merz, author of an excellent survey of 19th century science, that the vortex atom did not realize his expectations, inasmuch as it did not explain inertia or gravity. [232, p.1046] In 1898 he wrote Holman, professor at M.I.T. and surveyor of physics at the end of the 19th century: "I am afraid it is not possible to explain all the properties of matter by the Vortex-Atom theory alone, that is to say, merely by motion of an incompressible fluid; and I have not found it helpful in respect to crystalline configurations, or electrical, chemical, or gravitational forces. ... With great regret I abandon the idea that a mere configuration of motion suffices" [232, p.1047].

Kelvin's recognition of the failure of the vortex atom model led to a short refuge in the Boscovichean theory of matter (see §5.1). This seems remarkable as in the early 1860s he had completely rejected this theory. However, he must also have realized that the space-filling force of Boscovich's theory was not so very much different from the space-filling vortex ether theory. While in his 1884 Baltimore Lectures he had put the Boscovich model behind the vortex atom, in the 1890s he was readily employing Boscovichian force curves (see e.g. [281, Ch.9]). While he retained a speculative belief in the sufficiency of explanation by means of models like that of the vortex atom, he had shifted, as a practical matter, to the more positivistic approach of Boscovich. However, he still insisted that atoms had to be considered as having finite dimensions and structure, properties which the Boscovichean atom was lacking.

Finally, however, as for the vortex atom, Kelvin recognized that the Boscovichean atom could provide no sufficient explanation for matter. Kelvin's final atom model of 1901 [260] was a static arrangement, which contained electric particles which he called electrions.

## 6.3 The Rise of a New Physics

British physics after 1880 showed a complex interaction between Maxwell's electromagnetic theory, vortex ether theories, new insights into the nature of the electric charge, and modified vortex atom conceptions of matter. This "struggle" would eventually be lost by the vortex atom and all related models that had failed to incorporate the electromagnetic theory of light. This is apparent by the change in Kelvin's opinion on matter as we already indicated at the end of §6.2.

Several incentives towards the new development in physics, i.e. the shift towards "electric" models, can be mentioned.

One incentive was related to the second cloud over the dynamical theory of heat and light, which Kelvin had treated in [259] (see §6.1): the Maxwell-Boltzmann equipartition theorem (see §5.3.2). While Kelvin had tried to dispel its meaning and to show its failure, in the 1890s other British physicists incorporated the electric charge into vortex atom conception in an attempt to reconcile the difficulties related to the equipartition theorem.

Another incentive came from the still ongoing attempts to adapt the vortex atom to new developments in physics. Larmor, then lecturer in Cambridge and eventual successor at Stokes's position, attempted to resolve the problems of constructing an ether theory that would represent all optical and electromagnetic phenomena. In the first part of his extensive paper "A dynamical theory of the electric and luminiferous medium" of 1893-1895 [116], we encounter a man still fighting with the heritage of the declining vortex atom and the vortex ether models. However, Larmor came to the conclusion that some other bond for the atoms of a molecule had to be found, in addition to the hydrodynamic one. This he found in the attractions of the electric charges of the atoms. Thus, while the first part of his paper had been a last attempt to reconcile the vortex theory with the newly emerging concept of electrons, the second part would be completely devoted to the electrons.

Not surprisingly, Larmor's theory encountered the same attacks as the vortex atom theory: it was too complex and far-fetched to be real. Nevertheless, Larmor's work found some support in a paper by Pocklington [179](1895) who discovered that the missing energy in the model could be found in the electric charge of the vortex.

Though British physicists like Larmor were still trying to integrate Maxwell's theory of electromagnetic fields into Kelvin's ethereal continuum, others just formulated field equations without any involvement of (mechanical) ether models. The development of ether and field theories more and more challenged the hegemony of the view of nature expressed in e.g. the vortex atom theory.

The final blow to the vortex atom was given by J.J. Thomson, former defender of the vortex atom (see §5.1).

The *Treatise* had given Thomson some reputation. In 1884, at 28, he was chosen as Cavendish Professor of Experimental Physics at Cambridge University (Maxwell and Rayleigh had been his predecessors), above men like FitzGerald and Reynolds. Recalling the failures he had met in his elaboration of the vortex atom theory (as treated in §5.3), it is not surprising that after 1885 Thomson had growing doubts on the vortex atom and acquired a positivistic viewpoint on matter. After 1886 he stopped research on vortices and went over to experiments on rarified gases to verify his ideas on chemical combinations. Nevertheless his work was still based on his investigations on the vortex atom. His research with cathode rays showed that their long path lengths could only be explained by assuming a very small particle, very much smaller than an atom, which would be inconsistent with the vortex atom model. Even if Larmor's model would be used, Thomson realized, no explanation could be found for this phenomenon. However, the vortex theory still guided his search [266]. A theory of cathode rays based on linking and unlinking vortex rings led Thomson to think that the phenomena would be clearer in a higher vacuum. The final result of these experiments was the discovery of the electron in 1897 [151, p.173-]

Thomson's discovery meant a definite justification for the newly arising "electric" atom models. A new physics had been born. As Pearson accounted in 1900:

> The end of the nineteenth century ... marks the advent of experimental knowledge requiring an entire revision of the hypotheses and theories as to the constitution of matter. ... Whereas through the greater part of the nineteenth century, "matter" was the concept which was looked upon as fundamental in physical science, of which there was a curious accidental property called electricity, it now appears that electricity must be more fundamental than matter, in the sense that our once elementary matter must now be conceived as a manisfestation of extremely complex electrical phenomena. [176, p.356-]

Related to this shift in "fundamental concept", i.e. from mechanical towards electrical, the end of the 19th century saw the decline of devising mechanical and hydrodynamical models (see the introduction of Chapter 5). If the supposed mechanical explanation provided no new insight, and if it led to no further progress, in what sense did it provide an explanation? Planck's attack on the kinetic gas theory, for example, in the 1890s, can be seen as the growing challenge of the whole "mechanical programme". Even the position of matter was attacked. If all matter is made of elements which are of an electrical nature, is not electricity instead of matter the fundamental physical reality? It was the fate of the vortex atom model to introduce the theory of fields into Britain, which would result in field theory of the atom in which the vortex atom was no longer needed.

Larmor, who had seen the decline of his own model, tried to formulate the merits of the vortex atom model in his 1900 presidential address for the British Association, which can be regarded as a summary of the state of matter and ether theories at the turn of the century:

> The vortex-atom theory has been a main source of physical suggestion, because it presents, on a simple basis, a dynamical picture of an ideal material system, atomically constituted, which could go on automatically without extraneous support. The value of such a picture may be held to lie, not in any supposition that this is the mechanism of the actual world laid bare, but in the vivid illustration it affords of the fundamental postulate of physical science, that mechanical phenomena are not parts of the scheme too involved for us to explore, but rather present themselves in definite and consistent correlations, which we are able to disentangle and apprehend with continuously increasing precision. [117, p.625]

Kelvin not only had to accept that his theory of "matter as motion" had failed, it must also have bothered him that the new generation of physicists had stopped using his kind of models and had shifted towards analytical models [9]. Kelvin's methodology would die with him and with his vortex atom.

[9] See the Epilogue.

# Interlude: Between Vortex Atom and Vorton

As we have seen, the introduction of the vortex atom by Kelvin meant a new impulse to research on vortex motion. The impressive works by Thomson, Hicks, and several others (see §5.1) marked a transition towards a serious, mathematical treatment of vortex motion. Though the decline of the vortex atom may have caused a temporary stagnation in the development of vorticity theory , since Kelvin's days the subject of vorticity and vortices has been steadily enriched and and is still actively explored. Moreover, it has become generally recognized that vorticity is an essential part of most fluid flows. Several new topics in research on vortex motion have been introduced of which we mention, without further exposition, vortex-breakdown (see e.g. [205, §14.4]), geophysical vortex flows, vortices in wakes of solid bodies, vortex shedding, vortex buckling, vortex sound, and vortex merging [10].

As has become clear from §5.2, in the 1880s and 1890s research on vortex motion hardly spread from Britain to other parts of the world, and seems to have remained a British speciality for some time after the decline of the vortex atom and vortex ether. Only in the first decade of this century, a growing interest can be detected on the Continent.

In the beginning of the 20th century new areas in fluid mechanics developed, like the theory of boundary layer flow and the theory of airfoils [11]. Their popularity temporarily hindered recognition of interesting and important results which were still discovered in vorticity theory [12]. At the same time, however, discoveries in these fields showed new and unsuspected aspects of the role of vorticity in fluid flows.

In 1904, the German Prandtl proposed the boundary layer, a thin layer near the surface of a body in which vorticity is generated due to the so-called no-slip boundary condition [13]. Due to the realization that the viscous boundary layer could be regarded separately from the inviscid flow above it, adoption of his theory permitted mathematical simplifications of the hydrodynamical equations which resulted in the solution of some long-standing viscous flow problems in fluid mechanics (e.g. the drag met by bodies in fluid flows). The early developments in the design of airplanes gave an important stimulus to the field of aerodynamics. The new theory of airfoils, formulated by the Russian Zhukovsky and others, showed the important connection between the lift of a wing and Kelvin's (still underrated) concept of circulation. Besides, this new field stimulated the study of compressible (vortical) flows [134, §4.7].

In another major field of 20th century fluid mechanics, turbulence, developments at the beginning of the 20th century were still slow and few [14]. Only in the 1930s vorticity became involved (again) in turbulence research, as will be discussed in §B.

Here, we will not review all developments in vorticity theory up to our days [15]. The choice

[10] The only available general survey of vortex flows, both in nature and technology, is the book by Lugt [134]. A survey of present-day topics related to vortex motion is [13].

[11] We refer to [67] for a concise survey of research in fluid mechanics in the first part of the 20th century.

[12] A nice example in this context is the publication in 1906 by the Italian scientist Da Rios of the dynamical equations for the global behaviour of vortex filaments. This paper fell into complete oblivion, to be only rediscovered in the 1960s [192].

[13] See [230] for a history of boundary-layer theory.

[14] Lamb inserted one (short) section on turbulent motion in the second edition (1895) of his *Hydrodynamics* (see §5.1). This was kept unaltered up to the last, 6th, edition of 1932.

[15] Unfortunately, no (comprehensive) review seems to exist on the development of vorticity theory in the 20th

of the topics presented in this Interlude has been guided by the subjects to be treated in the vorton-part of this thesis. In §A developments in experimental and analytical treatment of vortex rings will be discussed. The last three sections contain an introduction into the role of vorticity in modern fluid mechanics: §B on turbulence, §C on topological fluid mechanics, and §D on vortex methods.

## A Vortex Rings

In this section we will present some of the 20th century developments in the experimental and analytical treatment of vortex rings. Although their use in theories of matter or ether had become almost completely abandoned by 1900, their recognition as the most fundamental (closed) vortex structure caused a continuing interest in their characteristics.

### A.1 Experiments

At the end of the 19th and the beginning of the 20th century, the essentials of experimental work on vortex rings were largely based on the first achievements by Tait [16] though the execution of the experiments became more sophisticated. During the 1870s, Ball, professor of applied mathematics and mechanics at the Irish Royal College of Science, had received a grant from the Royal Irish Academy to develop a machine for producing smoke rings. In his highly impressive experiment he studied the retardation of vortex rings due to viscous diffusion (see e.g. [19]).

In 1901, Wood, then instructor at the Physical Laboratory of the University of Wisconsin, published several experiments with smoke rings [285] and one of his pictures (see fig.a) showed the "fusion of two rings moving side by side into a single large ring. ... At the moment of union the form of the vortex is very unstable, being an extreme case of the vibrating elliptical ring. It at once springs from a horizontal dumb-bell into a vertical dumb-bell ... and then slowly oscillates into the circular form ...". This result seems to be the first description of an important aspect of vortex dynamics, which has only recently got more serious attention: vortex reconnection (see §C below).

In 1911, similar results though from a more sophisticated experiment on vortex rings were published by Northrup, then at Princeton University (see [166] for full details and [167] for a summary of his results). Introducing his investigations, he remarked:

> It seems strange ... that though the laws of vortex motion were exhaustively examined by the ablest mathematicians of the time, few if any experiments were made to study vortex motions in air and fluids, beyond the first experiments with smoke rings. ... The experiments which are about to be described would, if made earlier, have possibly had a greater interest as bearing upon Lord Kelvin's ingenious theory of the vortex atom. [166, p.213]

Northrup had constructed an ingenious "gun" with which he could make vortex rings in almost any initial configuration he wanted. He was even able to make beautiful stereoscopic

century. To get an impression of the development in this field, one could consult bibliographies like the *Royal Society of London Catalogue of Scientific Papers 1800-1900 (Cambridge, 1909)* and the annually published *Annalen der Physik und Chemie* and *Die Fortschritte der Physik*. For developments in the last three decades we can refer to the reviews which have been published in the *Annual Review of Fluid Mechanics*.

[16] An alternative method to produce rings was suggested by J.J. Thomson. In 1885 he presented results of experiments he had done together with Newall at Cambridge. They noticed that a drop of ink became unstable when it fell into water and secondary smaller rings developed [240]. This work on rings formed from liquid drops was only reconsidered in 1966 [34].

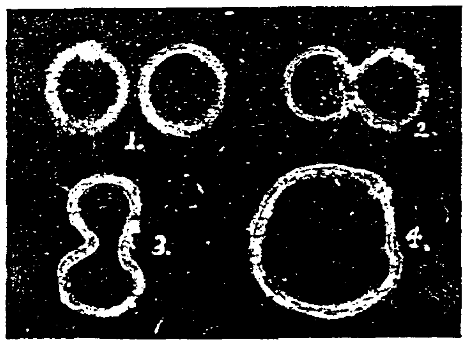


Figure a: Illustration of reconnecting vortex rings from Wood's 1901 experiment [285]. The rings are moving away from the reader.

photographs of the interaction of these rings. Besides experiments on elliptical rings and their instability, Northrup repeated Wood's experiment on the interaction and subsequent reconnection of two initially parallel rings. His sketch of the process (see fig.b) was accompanied by the following comment:

> In Fig.[b] *a* shows two rings a few centimeters from the gun. They are shown in side view, moving in the direction indicated by the arrow, *m*. The velocity of the fluid *in reference to the rings* is indicated at three points by the arrows 1, 2, $2^1$. Since this velocity is greater at 1 than at 2 and $2^1$, the points on the circumferences of the rings which are adjacent lag behind the points which are opposite, and consequently the planes of the two rings begin to tilt forward in the manner indicated.
>
> Furthermore, as the velocity of the fluid is greater at 1 than at other points equally distant from the filaments of the rings, here will be acting a pressure, according to Bernoulli's principle, which will tend to force together adjacent points of the circumferences of the two rings. Hence, a moment later the rings will assume a position indicated at *b*. The rings should now be viewed from behind, when their form will be (at a very brief instant later than shown in *b*) as indicated in full line at *c*, or in dotted line at *d*, which is a side view. The high velocity of the fluid at the point of contact has caused the two rings to unite and assume the form of a figure 8, with its upper and lower ends greatly tilted forward in the direction of motion of the now single ring.
>
> As this distorted single ring has everywhere an equal tension along its filamentary line, it tends to assume a circular form and lie in a plane normal to the forward direction of motion. But in changing its form to assume the circular plane ring form it overshoots this equilibrium position and assumes, as seen from behind, the form shown at *d* in heavy line. This double oscillation about the form of equilibrium now continues, and the ring advances ... [166, p.366-8]

It seems that only in 1939 the next extensive experiments on vortex rings were performed, in Germany by Krutzsch [112]. Krutzsch discovered that instability of rings led to a pattern of

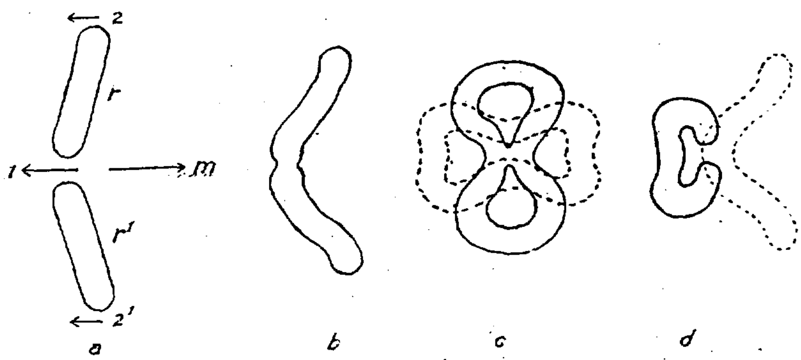


Figure b: A sketch of reconnecting vortex rings, drawn by Northrup in 1911 to illustrate his experimental results [166] (see text).

sinusoidal variations of the ring's radius, called azimuthal instability; see fig.c. The following war period seems to have prevented continuation of his work. Not before the 1970s experiments

Figure c: Instability of a vortex ring in the form of an azimuthal wave disturbance as found by Krutzsch in his 1939 experiment [112].

on vortex rings seem to have been resumed. Although the equipment had become even more sophisticated, researchers were still mainly concerned with the classical configurations: the head-on collision of two vortex rings (as Helmholtz had already discussed; see Chapter 2), the reconnection of two initially parallel vortex rings (as had been done in the experiments by Wood and Northrup), and the leap-frog effect. The latter, which had already been predicted by Helmholtz in 1858, was first demonstrated experimentally in 1978 [287] [17].

## A.2 Analytical Treatment

The analytical work on vortex rings by Kelvin and others in the last three decades of the 19th century had been impressive and their results were generally regarded as physically well-

[17]Discussion of more recent experimental results will be postponed to Chapter 10, where the simulation of six vorton configurations is treated.

founded for several cases. However, in their derivations the following properties of the rings had been assumed (compare fig.2.3):

- uniform vorticity distribution in the core and no vorticity outside the core;
- circular core;
- core radius $a$ small compared to the ring radius $R$: $a/R \ll 1$.

A vortex ring satisfying these conditions will be called a **Kelvin-ring** in this thesis. One started to realize that in most circumstances these "ideal" rings would be only very approximate representations of real vortex rings [18]. However, mathematical techniques were lacking for fuller treatment and the investigation of vortex rings whose conditions differed from those of the Kelvin-ring only started in the 1970s (see e.g. [276] and [205] for a survey).

One of the most elementary results on the vortex ring had been its velocity, for which Kelvin's expression (4.3) became a landmark. Hicks, who had been one of the most dedicated followers of the vortex atom theory and the main propagator of the hollow vortex ring (see §5.1), had confirmed the correctness of expression (4.3). However, he also showed that for his hollow ring (see §5.1) the factor $\frac{1}{4}$ had to be replaced by $\frac{1}{2}$, thereby providing a direct proof of the influence of the vorticity distribution in the core [82]. Further analytical confirmation of Kelvin's result appeared in an impressive paper of 1893 by Dyson, communicated by J.J. Thomson [54]. Dyson extended the expression to rings of non-circular cross-sections and found a higher order error estimation of expression (4.3). In 1914 Gray again confirmed Kelvin's result [69], but only in 1970 the velocity of a ring of small cross section with arbitrary distribution of vorticity within the core was derived for the first time (see [276] and [205] for details). For an arbitrary distribution of vorticity in the core (but still a small and circular core), it was found that the ring velocity could be written as:

$$V = \frac{\Gamma}{4\pi R}(\log\frac{8R}{a} + A - \frac{1}{2})$$

where the factor $A$ depends on the vorticity distribution only. For uniform vorticity $A = \frac{1}{4}$, confirming Kelvin's result [19].

The attempts by Thomson in his *Treatise* of 1883 (see §5.1) had shown the limitations of analytical treatment of the interaction of vortex rings [20]. Subsequent analytical research had necessarily been limited to the relatively simple cases of head-on collision and leap-frogging of coaxial rings. Thomson's followers also realized that only configurations of coaxial vortex rings

[18] In 1888, Chree [38] had already shown that cores of vortex tubes may not remain circular.

[19] J.J. Thomson in his *Treatise* (see §5.1) had proposed a factor 1 [234], but Hicks convinced him that he was wrong [82]. Experimental confirmation of the above equation for $V$ was only tried by Sullivan *et al.* in 1973 (see [276]); they found differences within 20% of the theoretical values. In the last few decades expressions have also been found for unsteady rings, for compressible rings, viscous rings, and rings with swirl; see for a discussion [205, §10.3].

[20] Roberts [193] took up J.J. Thomson's analytical work on interacting vortex rings in 1972. He remarked that the interaction of rings are "strongly reminiscent of those given in standard texts for scattering under central forces", which suggested to him that it could be described from the standpoint of classical Hamiltonian dynamics of interacting particles. However, Roberts noticed that rings cannot be compared to elastic particles since a ring has an infinite number of degrees of freedom (compare Maxwell's remark mentioned in §5.3.4). However, if the separation of the rings is large compared to their diameter, collision is elastic and for this case Roberts presented a Hamiltonian formulation. He showed that J.J. Thomson had made an error in his derivation, which did destroy the Hamiltonian character of his final results.

would be feasible for analytical treatment. Dyson's paper (see above) contained fundamental results on the leap-frog interaction and on the head-on collission of two Kelvin-rings. For the latter, he (like Helmholtz in 1858) recognized that it was equal to the interaction of a single ring with a "fixed plane"; in this case the ring interacted with its "mirrored" image on the other side of the plane. By means of impressive calculations Dyson was able to provide an exact analytical expression for the trajectory of the core center of the ring approaching the plane. Besides, he derived an equation for the rate of change of radius $R$ when the ring had approached the plane closely:

$$\dot{R} = \frac{\Gamma}{2\pi R}$$

where $\Gamma$ is the circulation. For the relation between $R$ and the distance $d$ between the ring and the plane, he found the curve shown in fig.d. For leap-frogging rings he derived conditions for

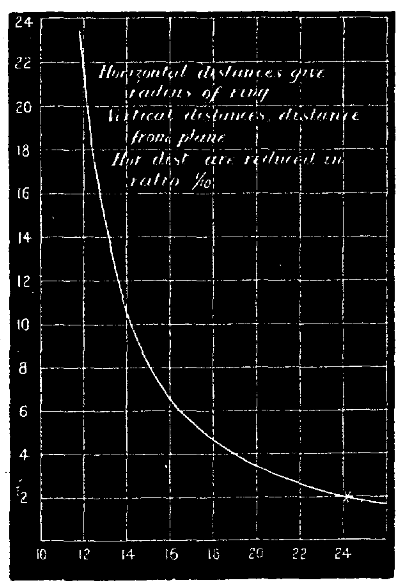


Figure d: Dyson's figure showing the relation between the radius $R$ (*horizontal*) and distance $d$ (*vertical*) of a vortex ring approaching a "plane". From [54].

the unbounded continuation of the process. In 1922, Hicks [88] would also investigate the leap-frog interaction and head-on collision. However, his work once again demonstrated the strong restrictions imposed by the available analytical techniques and the need for new methods to handle vortex interaction. Only the advent of numerical techniques would make an important progress on this point possible (see §D below).

### A.3 Steadiness and Stability

As discussed in §5.3.1, proof for the steadiness and stability of the vortex atom became an important issue in the discussion of the model. Regarding steadiness, only attempts by Kelvin himself have been traced and his results on this issue, as on other ones, lack sufficient rigour. Apparently, a real proof for the existence of steady Kelvin-rings was only given by Maruhn in 1957 and Fraenkel in 1970, though the question remains a subject of study (see e.g. [7]).

Regarding stability, in 1875 [248] Kelvin was still convinced that the Kelvin-ring was a steady and stable configuration. By 1905, however, he had changed his opinion and realized that it was "essentially unstable" (see §6.2). His main attempt with regard to this issue had been his 1880 paper [250] on the stability of "columnar vortices" by means of a dispersion relation for linearized perturbations [21]. However, he had to admit that he was unable to give a proof [22]. Thomson had tried to show that up to $O(\log a/R)$ all modes of oscillation are stable and that each mode had a definite natural frequency [234, §13].

Others also came to recognize that no convincing proof existed on the stability of vortex rings. In 1885, Lodge [130] analytically investigated a ring moving in a "oblique direction" and found "a good deal of vibration, both of the ring as a whole and of its cross section; and it looks as though a very little would suffice to break it up altogether." In his lectures on vortex theory of 1893 (see §5.2), Poincaré showed that there were no sufficient guarantees for the stability of the vortex atom [180].

As for vortex ring experiments, new investigations on their stability only reappeared in the 1970s. Maxworthy [149] has showed experimentally that a laminar ring is only stable at low Reynolds number [23] though stability of the ring configuration seems to set in again when the core becomes fully turbulent. The instability of the laminar ring takes the form of bending waves around the perimeter, and these waves grow in amplitude as time proceeds. Some of these "steady modes" had already been drawn by Kelvin in 1875 (see fig.5.1) and had been observed in the experiment by Krutzsch (see fig.c). Maxworthy attributed this instability to the method of generation of the rings and did not regard it as an intrinsic property.

However, the analytical work by Widnall and co-workers on the stability of vortex rings (or locally curved vortex filaments in general) showed that even for inviscid rings instability can arise regardless of their generation (see [277], [278], [279]). They showed that the proper treatment of the internal structure of the flow within the core as influenced by bending waves is crucial to stability analysis. This was something which both Kelvin in his investigation of the columnar vortex [250] and Thomson [234] had not considered [24].

Notice, however, that Widnall's analysis is based on the Kelvin-ring. Studies on the stability of more general vortex rings representations, e.g. with nonuniform core distributions, are still lacking and may only be possible by means of numerical methods (compare §10.1.2 below).

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[21] It appears that Kelvin's work was experimentally investigated only in 1989 by Vatistas [272]. He found unstable transitional regions between equilibria, which Kelvin had not noticed.

The perturbations studied by Kelvin are nowadays called Kelvin waves [205, §11.3]. The variational principle proposed by Kelvin in [248] has again been applied in the 1980s [205, §14.2].

[22] See [194] for a recent review of Kelvin's approach.

[23] The Reynolds number can be defined by:

$$Re \equiv \frac{UD}{\nu}$$

where $U$ is a characteristic velocity, $D$ is a characteristic dimension, and $\nu$ is the kinematic viscosity. For its history, see [200]. For vortex structures, another definition of Re can be found: $\text{Re} = \Gamma/\nu$, where $\Gamma$ is the circulation.

[24] Ironically, Kelvin's analysis for the neutral waves can be used as one of the ingredients to a simplified demonstration of the instability mechanism [212].

Recently, Lifschitz in [33] investigated stability by another method. His geometrical optics approach allows to describe short wavelength instabilities, which "play an important role in many situations". Whereas Widnall's theory was only applicable to thin rings with constant vorticity in the circular core, Lifschitz could study thicker rings. He showed that all laminar rings are unstable.

## B Vorticity and Turbulence

Without doubt, at the time of the birth of the vortex atom (1867) the phenomenon of turbulence had already been encountered in many flow configurations, but it certainly had not become an object of study. In 1883, Reynolds [191] had been the first to investigate by experiment the transition from laminar to turbulent flow and his work meant an important stimulus to research on turbulent motion [25]. Kelvin's 1887 paper on a possible model of the vortex ether [256] (see §6.1) had originally been entitled "On the propagation of laminar motion through a turbulently moving inviscid liquid" and had been an attempt to investigate "turbulent motion of water between two fixed planes" [26]. In 1889, FitzGerald [59] reacted to Kelvin's paper by an investigation of turbulent motion and possible analogies with electro-magnetic equations. He remarked that desintegration by diffusion of turbulent flow (as represented by Kelvin's model shown in fig.6.1) could be avoided "by supposing the turbulent liquid to consist of interlocked vortex rings, or of infinite intercrossing lines ..." [59, p.34] [27]. However, Kelvin's crude vortex ring model of a turbulent motion was not based on any physical knowledge of turbulence, which around 1887 was still very limited and mostly experimental.

This connection between turbulent motion and vortex motion followed the fate of the vortex ether itself and dropped from the general attention at the beginning of this century. Subsequent attempts to study turbulence from the general equations of fluid mechanics showed that the mathematics needed for full (analytic) treatment of turbulent flow was too difficult to handle, and approximations had to be made. This necessitated the use of physical insight. Prandtl's mixing-length model, introduced in the 1920s, was based on an analogy with the kinetic gas theory [28]. However, the prewar period was mainly a period of mathematical modelling and of experiments. Only in 1932, Taylor [231, Vol.II,§24] developed a theory in which the dynamics of turbulent motion was regarded as an effect of diffusion of vorticity. He showed that his suppositions led to better agreement with experimental results than the diffusion of momentum theories popular at that time [29]. In 1938 [231, Vol.II,§41], he showed the importance of the stretching of vortex filaments in turbulence.

Regarding contemporary literature on turbulence, we can conclude that Kelvin's and FitzGerald's original idea of representing turbulent motion by arrays of vortex rings has returned. Already in 1943 a similar attempt at modelling turbulent flow by restricting attention to the influence of vorticity was made by Synge & Lin [225]. They tried to derive the statistical characteristics of turbulence from a model consisting of interacting vortices. Their initial choice for vortex rings, however, had "undesirable features" and the authors turned to a model involving "spherical vortices". Since this attempt, many researchers have been incited to building turbulence models in which the basic concept is some kind of vortical structure [30].

[25] Unfortunately, as for the theory of vortex motion, no historical survey of the early history of turbulence research has been traced.

[26] Kelvin seems to have actually introduced the term "turbulence" [115, 4th ed.,§366]. Reynolds, in 1883, had spoken of "sinuous flow".

[27] The idea of representing turbulent motion by means of a system of vortex rings has been a topic ever since. One example is Roberts's model [193], mentioned above. More recently, Aref & Zawadzki in [16] wondered whether turbulence can be described as a "gas" of vortex rings.

[28] Notice that this meant a shift in the use of analogy in turbulence research from the vortex atom theory to the rivalling kinetic gas theory.

[29] Nowadays, this diffusion theory is no longer accepted.

[30] One of the latest models is by Lundgren [136], to whom we also refer for a discussion of preceding theories. One of the central issues in the investigation of "vortex models" has been the quest for the correct energy spectrum. The energy cascade at high wave numbers is believed to be independent of viscosity, so the classical

The purpose of the models mentioned above has been to obtain a simplified view on turbulence and to derive some of its essential properties [31]. The vortical elements were not supposed to be physical models of real structures in turbulent flows. Today, however, it has become clear that vortical structures are indeed present in turbulence. To elucidate this development, we will first give a short historical review.

In the 1930s the main treatment of turbulence, regarded as a random fluid motion, had become statistical. However, during the 1940s and 1950s the suggestion arose that besides the random part also a non-random part existed. A growing amount of experimental results, amongst others the observation of the so-called intermittency, led to the idea of the existence of "structures" in turbulent motion. Moreover, these so-called **coherent structures** (CS) were defined as regions of relatively high vorticity. Hence turbulence became envisaged as a number of interacting vortex structures. Especially in transitional flows, vortices were supposed to play an important dynamical role. By the 1970s this view of turbulence had become generally accepted [32]. However, it also became clear that, as Betchov in [60] remarked, "it is not the mere presence of vorticity that characterizes turbulence. It is the complexity of the vorticity field. In a laminar boundary layer, the vortex lines are parallel and stacked near the wall, like uncooked spaghetti. In the turbulent boundary layer the vortex lines are constantly changing and twisting. Near the wall, major entanglements appear, and the vortex lines may develop knots and crossover points. The spaghetti is cooked."

Although nowadays the existence and importance of CS is generally recognized and has been investigated both experimentally and numerically, still consensus is lacking on many aspects [33]. Several different structures have been proposed, but limited quantitative evidence hinders demonstration of the role these structures play. According to Kline & Robinson in [73], three main issues in the present research on CS can be detected: "spatial relationships among the forms of structure; temporal relations in creation, evolution, and decay of structures; a complete model of the important structure(s)".

Even a generally acceptable definition of CS still seems remote and may even be unattainable. Besides, it is still debated whether CS are the remnants of some kind of instability process or whether they are manifestations of some intrinsic universal properties of any turbulent flow. Up to now, research on CS has been done only for transitional or rather low Reynolds number flows and the question has been raised whether CS will survive in "fully developed" turbulence, i.e. at high values of Re. Unfortunately, research on CS is hindered by the difficulties involved in the direct measurement of vorticity in a flow. In numerical research the main problem is a lack of detection methods of CS [196, Ch.9]. However, it is generally agreed that both experimentally and numerically the importance of CS has been established and for the moment we will therefore disregard these problems.

On the close link of CS with vortex dynamics, we quote from the contribution "Whither coherent structures?" by Bridges *et al.* in [135]:

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Kolmogorov 5/3-spectrum may be obtained from interaction of inviscid vortex filaments. Kiya & Ishii applied an inviscid vortex method to show how only a few vortex rings, arranged symmetrically on a cube, can produce a Kolmogorov energy spectrum [105]. Moffatt, in [135] and [16], has proposed another model to attain the same results. He suggested that a random distribution of spiral structures (rolled-up vortex sheets) shows a Kolmogorov spectrum.

[31] See the Epilogue for a discussion of models.

[32] See [196, Ch.2] for a historical review of turbulence structure experiments.

[33] The literature is overwhelming and we only mention [92]. A more recent topic in turbulence, related to that of CS, is the study of the structure of vorticity in (isotropic) turbulence; see e.g. [95].

> What troubles us most is our inability to embody information gleaned from the experimental studies of CS into a mathematical framework. We feel that providing a mathematical basis for the CS concept will be the topic of considerable effort for years to come and will bring about a much better *understanding* of turbulence [i.e. explanation, prediction, control]. ... Given the topology of the vortical CS we can say roughly how it will evolve and interact with other CS. This is an advantage not held by other specific definitions of CS. Vortex dynamics gives local, short-term predictability of the dynamically important aspects of the flow. ... If CS are defined by vorticity, their evolution and interactions are directly connected to their topology through vortex dynamics. This is why it is important to categorize CS morphologically. ... Vortex dynamics is the missing mathematical framework for the study of CS. ... Using vorticity to define CS also allows us to predict flow evolution in complicated flow situations using intuition ... instead of having to resort to direct calculation ... . [135]

Although this view on the role of vortex dynamics in research on CS seems reasonable, we should also realize that the deformation and interaction of CS may well be a much more complicated matter than in case of "ordinary" vortex structures like vortex rings. For example, Melander *et al.* in [216] have mentioned five categories of close CS interaction: self-deformation of a single isolated CS, including effects of turbulence it may generate; interaction of a single isolated CS with background potential; interaction of a single isolated CS with turbulent background; isolated interaction of two CS in very close proximity; and isolated interaction of two CS in the presence of a turbulent background. To which they remark: "Only a thorough insight into the dynamics of key vortex interactions can further the present level of understanding of turbulence and the role of CS."

## C Vorticity and Topological Fluid Mechanics

Kelvin in his paper "On vortex motion" (see §4.1) was the first to perceive dimly the bridge between mathematical topology and classical fluid mechanics. In this paper he had explained how to avoid the consequences of multiply continuous spaces [34], probably intending to show the possible existence of the many varieties of the vortex atom; see fig.4.1.

Induced by Kelvin's still primitive results, Tait decided to classify and catalogue all knots of increasing order of complexity [227](1876). This catalogue of knots remains as the cornerstone of knot theory, now a well-established branch of topology [35].

Only recently, the link between topology and fluid mechanics has resulted in the field of "topological fluid mechanics", which today seems to be a firm branch of fluid mechanics (see e.g. Moffatt's lecture in [160] and [162]) and has partly been stimulated by the research on CS.

In this new field an important role is played by the so-called helicity field. In the 1960s Moffatt [157] discovered that the topology of vortex structures is closely linked with one of the

[34] Kelvin adopted terminology introduced by Riemann, known to him through Helmholtz. Although the theory he developed here has never been applied to the vortex atom theory or any related subject, Kelvin's solution of the problem (which had been posed by Helmholtz) of extending Green's theorem to multiply-connected regions was certainly a high mathematical record of this paper. His proof technique is still used today, which shows an unsuspected heritage of this paper.

[35] See Millett in [162] for a historical account of the development of knot theory and Tait's role.

motion-invariants of the Euler equation, i.e. **helicity**:

$$H \equiv \int_V \boldsymbol{w} \cdot \boldsymbol{v}$$

where $V$ is the volume in which the fluid flow under consideration is taking place. Conservation of helicity means that the "topology" of vortex structures

in the flow remains unchanged, something which Kelvin had already recognized in his paper "On vortex motion" of 1869. In viscous flows changes of topology can take place, and $H$ is no longer a motion-invariant. Helicity may also be of profound importance in the non-linear *dynamics* of turbulence, e.g. to characterize organized or coherent motion in turbulent flows. This supposition has been extensively studied and propagated by e.g. Levich [124], but others found opposing results (see e.g. [275]). Lack of experimental results hinders reliable opinions or conclusions [36].

One important aspect of modern research in the topology of vortex structures concerns vortex reconnection.

The experiments performed by Wood and Northrup at the beginning of this century (see §A.1 above) had already shown that vortex rings may become united into a single structure. Today this phenomenon is indicated by the terms cut-and-connect or **reconnection** [37]. However, the importance of vortex reconnection, of which the linking of vortex rings is one example, was not realized until many years later. During the last few decades, as computer technology and numerical possibilities rapidly advanced and new aspects of reconnection were discovered, interest grew quickly [38].

It is now generally accepted that vortex reconnection is only possible when viscous diffusion of vorticity is present [39]. See fig.e for a elementary sketch of the reconnection process and compare with Northrup's sketch in fig.b. In the initial stage of the reconnection, vortex tubes

[36] We refer to [161] for a review of helicity in fluid mechanics.

[37] We will stick to the use of the term "(vortex) reconnection" throughout this thesis, but recognize that the term "cut-and-connect" sometimes is a better description of the phenomenon discussed.

[38] An interesting revival of the use of analogies between hydrodynamical and electromagnetic phenomena, initiated by Helmholtz and Kelvin, can be detected in the present interest of fluid mechanicists in some parts of research from magnetic field theory, where helicity and reconnection have also become familiar concepts.

Berger and Field [24] showed that magnetic helicity is also closely associated with many aspects of topological structure of the magnetic field. In plasmas with high but finite magnetic Reynolds number, it has been conjectured that reconnection of field lines can alter the field topology, while approximately conserving helicity, as in fluid mechanics. This suggests that helicity is not a good indicator for change in topology.

Greene in [160] has applied mathematical techniques from the theory of magnetic reconnection, "perhaps slightly better understood" than vortex reconnection. However, Hegna & Bhattacharjee, also in [160], notice that for their treatment of magnetostatic equilibria, the analogy between magnetic fields and Euler (i.e. inviscid) flows is only valid up to a certain point.

[39] The theory of viscous vortex motion had already started during the time of the vortex atom, initially because some had argued that the theory was based on inviscid flow, whereas the experiments obviously produced viscous vortex rings.

In 1879 [233] Thomson had treated the vortex equations for viscous fluids and remarked the analogy with the equation for the conduction of heat. He concluded that any vortex motion in an initially irrotational flow must come from the boundary of the fluid: "if vortex motion be set up in any part of a viscous fluid, the motion throughout the fluid immediately becomes rotational". Reynolds [188] had observed that contrary to Kelvin's inviscid picture, the volume of the "vortex ring bubble" continually increased due to entrainment of external irrotational fluid and its velocity decreased because its momentum has to be shared with a greater mass of fluid. Lodge in [130] (see §6.1) had shown analytically that a viscous ring showed a decrease in velocity and an increase in size.

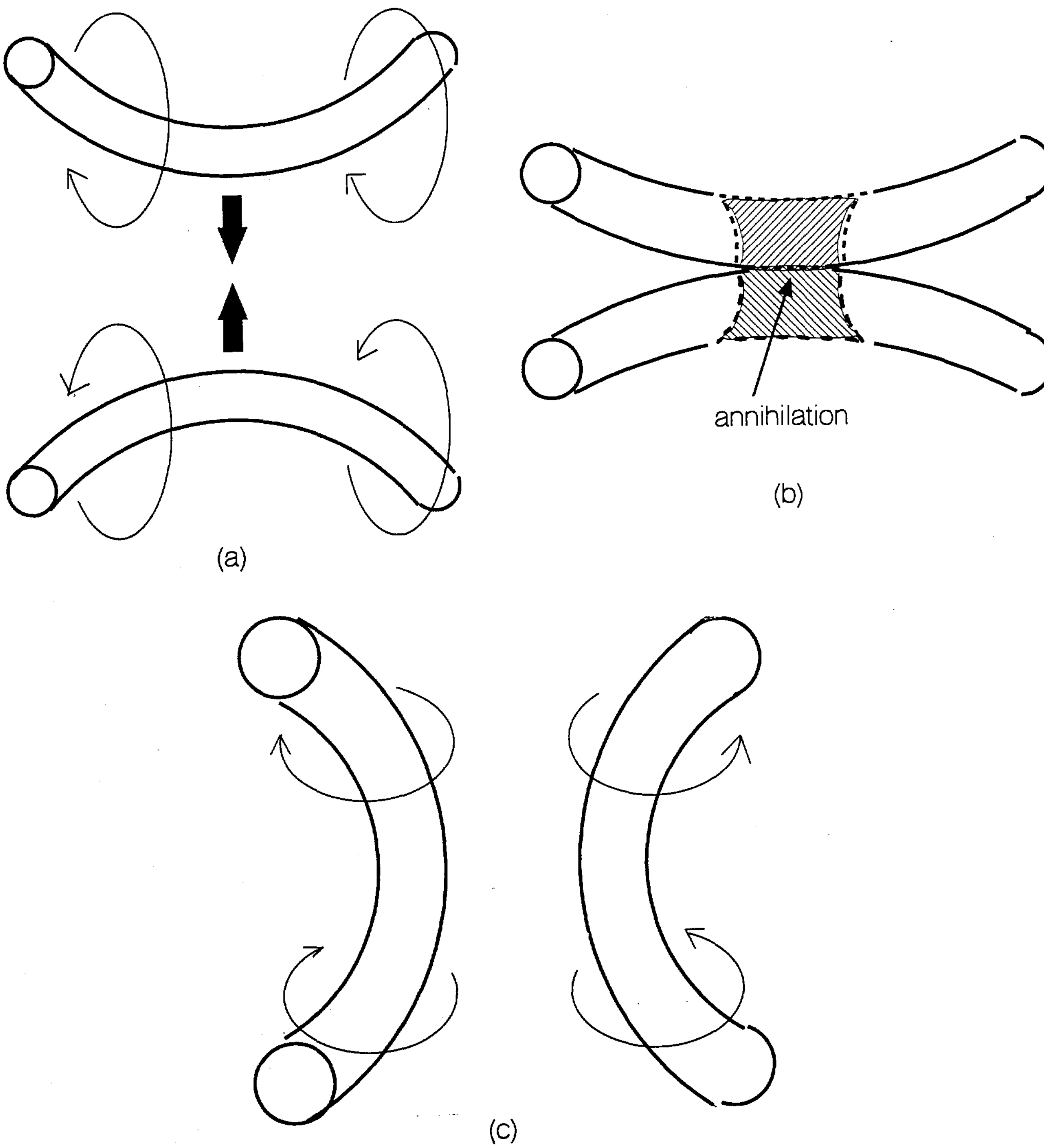


Figure e: Reconnection of two vortex tubes: (a) Alignment of oppositely directed vorticity; (b) Annihilation by viscous diffusion of oppositely directed vorticity in anti-parallel aligned vortex tubes; (c) Formation of new connections (bridging).

of oppositely directed vorticity show **alignment**. Alignment is the tendency of approaching vortex tubes which initially are positioned in a random orientation with regard to each other, to become aligned with the direction of the vorticity vector in both tubes directed anti-parallelly; see fig.f. Then, annihilation of vorticity takes place by diffusion. In the third stage new

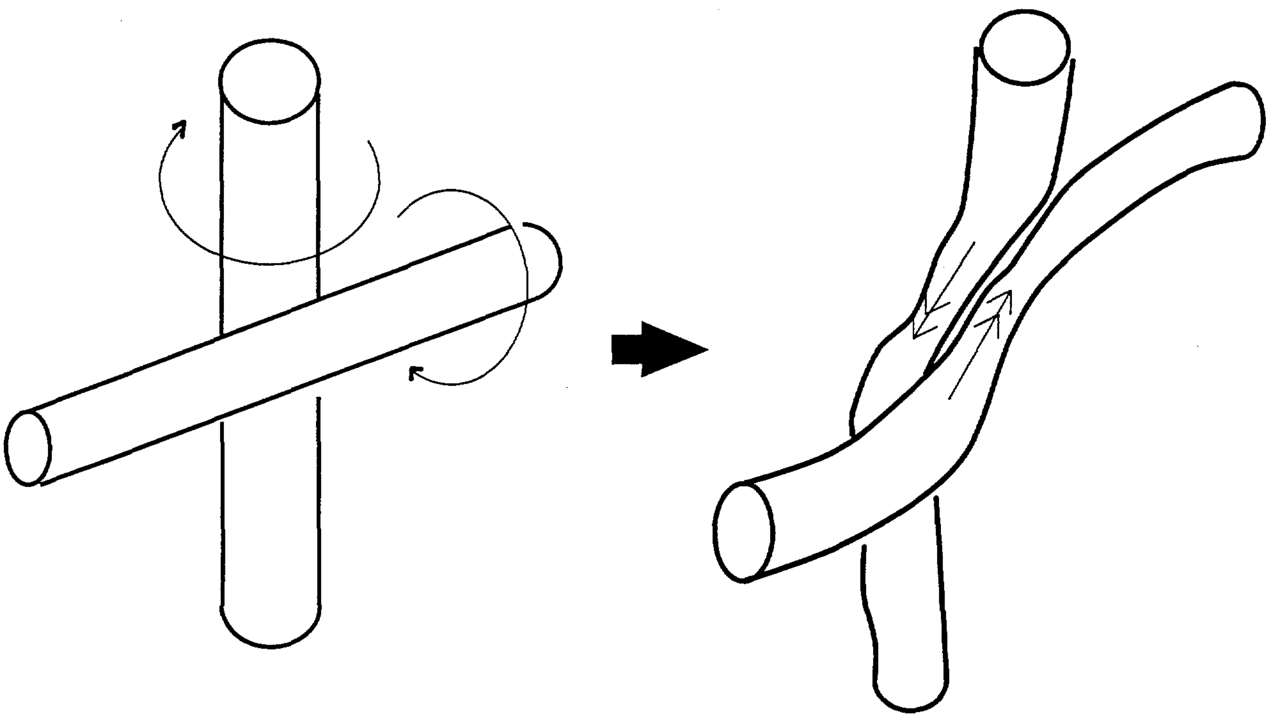

Figure f: Anti-parallel alignment of vortex tubes. Arrow indicates time development. Double arrows indicate direction of vorticity.

connections are formed, a process which has been called **bridging**. Two of the main aspects of reconnection which have been generally recognized are the deformation of the core of the vortex structures and the influence of local strain on the reconnection process. Recent research suggests that reconnection results in a rapid local increase of the strain rate and vorticity. Shelley & Meiron in [10] have suggested that core deformation plays an important role in the process. For example, it has been shown that deformation of the core effectively prohibits unbounded growth of vorticity during reconnection.

However, the exact mechanism of reconnection remains unclear. Most research on this phenomenon has been done by means of numerical simulation of generic test cases: two segments of vortex tubes, placed either orthogonally or anti-parallel (see e.g. [27]).

Analytically, reconnection is hard to treat. The only analytical model known is the one by Saffman [204], who found some agreement with experimental data, but had to admit that

Due to an early paper by Bobylew of 1873 [26] it was realized that whereas Helmholtz's first theorem was valid for real flows, the second theorem was not and vorticity could both be generated and destroyed by viscous effects, which led to discussions on the origin of vorticity in initially irrotational flows [134, §4.6]. Furthermore, it was realized that in viscous flows circulation was not preserved and the Helmholtz equation (2.2) had to be adapted to take viscous diffusion into account.

The study of viscous vortex rings seems to have been only taken up in the early 1970s, especially by Maxworthy [148].

some large discrepancies exist and that the model contains "questionable choices". Analytical work by Takaki & Hussain (see [13]) showed that reconnection takes place within a time scale of the order of the convective time scale rather than of the diffusive time scale [40]; nevertheless, they concluded that viscosity is necessary in the process.

Some have suggested that vortex reconnection is an important aspect of turbulence. It has been proposed as the mechanism for isotropization, for the production and dissipation of vorticity in the flow, for the so-called energy and enstrophy cascades, and for the production of helicity. However, on this issue opposing views exist. Ashurst & Meiron [18] speculate that reconnection may occur in a turbulent flow whenever two opposite-signed vorticity regions approach each other, whereas Boratav & Zabusky remark in [162] that it has not been proved whether reconnection really takes place in turbulent flow fields at all.

Unfortunately, since reconnection may occur randomly in space and time in a turbulent flow, it is hard to investigate experimentally or by means of direct numerical simulation (DNS). Therefore, our understanding of reconnection in turbulence is still limited and scattered [41].

## D Vortex Methods

In one of his *Lectures on some recent advances in physical science*, first published in 1876 [229] (see §5.1), Tait noticed the enormous mathematical difficulties involved in the elaboration of the vortex atom model. As an example, he thought that the interaction of two rings positioned non-symmetrically about an axis "may employ perhaps the lifetimes, for the next two or three generations, of the best mathematicians in Europe; unless, in the meantime, some mathematical method, enormously more powerful than anything we at present have, be devised for the purpose of solving this special problem" [229, p.302]. The attempts by Thomson in his 1883 *Treatise*, and by some others (see §5.1), indeed showed that analytical treatment of this topic could only be achieved if severely restrictive assumptions were made.

One way to avoid unsurmountable mathematical difficulties is a simplification towards two-dimensional (2-D) flow. Kelvin himself had already realized that for the demonstration of the stability of single or interacting vortex atoms, a reduction of the configuration towards infinitely long, straight, and constantly parallel "columnnar vortices" would improve the possibility of analytical treatment (see §5.3.1). In [249] he had discussed regular configurations of such columns, apparently without knowledge of the work that the German Kirchhoff had done some years before [42]. Kirchhoff had shown that the dynamics of these vortices, assuming an infinitesimally small core size, could be described by a Hamiltonian set of equations. In the second half of the 19th century, this subject was taken up by several contemporaries, e.g. by Gröbli [14].

The theory of these essentially parallel vortex filaments, or point-vortices as the 2-D cross-sections of these filaments with a flat plane perpendicular to their axes are called, came to be used in the 1920s and 1930s during the first attempt to solve flow problems by means of discretization of a continuous vortex structure. In 1931, Rosenhead [199] approximated a 2-D vortex sheet [43] by means of point-vortices, and in this way he was able to simulate the roll-up of such sheets. This attempt can be called one of the very first attempts to apply a

[40] This timescale is related to a typical convective velocity in the flow.

[41] See Caflisch in [32] and Boratav & Zabusky in [162] for a short review.

[42] See for references e.g. [14]. In his 1858 paper Helmholtz had also shortly treated the theory of parallel vortex filaments [75, §5].

[43] A vortex sheet is essentially a surface formed by vortex lines. It had already been described by Helmholtz in his 1858 paper, as Rosenhead remarked.

vortex method, i.e. to simulate the dynamics of vortex structures by means of discrete vortex elements. This enabled the investigation of vortex configurations which experimentally and analytically had been beyond any hope for solution.

However, vortex motion in two dimensions is essentially different from that in three dimensions. As can directly be seen from the Helmholtz vorticity equation (2.2), 2-D vortices do not deform. As it was realized that vortex stretching was an important mechanism in vortex dynamics, interest in the treatment of 3-D vortex motion started to draw attention.

Tait's "enormously more powerful" mathematical method which would finally bring new progress (both for 2-D and 3-D vortex dynamics) was the application of numerical techniques to solve the vorticity equations on computers. The advent of powerful computers in the 1970s made the use of so-called vortex methods a wide-spread tool in fluid mechanics with which the research into interaction of vortex structures (such as vortex rings) could be and indeed was revitalized. Not only did they enable the investigation of finer details of the interaction, also configurations could be investigated which had been inaccessible by experimental means.

The general set-up and a short survey of vortex methods will be the subject of the next chapter.

## Chapter 7

# Vortex Methods

During the last few decades, the availability of powerful computers has resulted in a completely new branch of fluid mechanics: computational fluid dynamics (CFD). Today, CFD has become an enormously broad field of research in which a great diversity of numerical techniques is used. Vortex methods form one part of this field and several widely differing examples have been introduced in fluid mechanics literature. In this chapter we consider general set-up of 3-D vortex methods, some of the requirements they should satisfy, and some of the pros and cons of some vortex methods. We also give a short survey of some classes of methods in order to point out several drawbacks of existent vortex methods [1].

## 7.1 General Set-up of Vortex Methods

Vortex methods are computational methods aimed at the simulation of fluid flows by following the evolution of their vorticity field. The vortical structures in the flow field are represented by so-called vortex elements whose behaviour follows from solving their dynamic equations numerically.

The existence of vortex methods is related to the fact that fluid flows can be described by means of equations like the Helmholtz equation (2.2). The use of this vorticity formulation of the hydrodynamical equations has several important advantages above other formulations, e.g. the velocity-pressure formulation of the classical Euler equation [2]:

- vorticity, contrary to e.g. velocity, is frequently localized in space; consequently the amount of computational elements to be taken into account can be kept relatively small [3];

- according to Helmholtz's second theorem, vorticity behaves materially; consequently a Lagrangian approach is attractive, i.e. we can follow the vorticity field by tracking vortex structures which move along with the flow;

- since reconnection and intermittency in turbulence (see §§B and C of the Interlude) occur randomly in space and time, these phenomena are hard to investigate experimentally or by means of direct numerical simulation (DNS); simulations by means of vortex methods may be more appropriate.

Besides these general incentives to use vortex methods, in case of turbulent flows the generally recognized importance of coherent structures and their presumed relation with vorticity (see §B of the Interlude) has been another important motivation for the use of vortex methods.

[1]For a general review of vortex methods, see e.g. [121] and [207].

[2]Notice that the Helmholtz equation (2.2) can directly be obtained from the Euler equation by applying the "curl"-operator.

[3]Related to the above characteristic of the vorticity field, it has been thought that, though fluid motion itself has an infinite number of degrees of freedom, a study of vorticity fields may only require a finite number.

The general set-up of every vortex method follows from answering the following three questions:

1. **How do we discretize the vorticity field? or: What are the vortex elements?**

   Several vortex elements have been proposed, both of infinitesimal and finite dimensions. The simplest elements are vortex points, which just have a location and a strength. The so-called vortex blobs (or smoothed vortex points) are characterized by additional **smoothing functions** which determine the (variable) shape and size of the vorticity distribution in these elements. Another element is the vortex filament (see the definition in Chapter 2), which is determined by its location, torsion and curvature, and by its strength. Again, like vortex points, vortex filaments can be smoothed.

2. **Which equations for displacement and deformation of vorticity do we use and how do we transform these equations into equations for the chosen vortex elements?**

   For the calculation of the displacement of the vortex elements, the following expression may be applied:

   $$\frac{D\boldsymbol{x}}{Dt} = \boldsymbol{v}(\boldsymbol{x}, t) \tag{7.1}$$

   where $\boldsymbol{x}$ is the location vector and $\boldsymbol{v}$ the velocity vector at this location.

   The velocity field can be derived from the vorticity field according to the rule of Biot-Savart (2.3). However, application of this rule may require a method to avoid the singularity in the kernel of this integral equation (so-called regularization). In case of smoothed vortex elements, this problem is avoided as a result of the finite "core" size. Another approach is the so-called cut-off of the integration interval of the rule of Biot-Savart [4].

   The deformation (rotation and stretching) of vorticity can be described by either the Cauchy vorticity formula (1.4) or the Helmholtz equation (2.2). The first is generally applied in Lagrangian methods (defined above). The second are used in so-called Eulerian methods, i.e. methods in which the vorticity is regarded at fixed locations in the flow.

3. **How do we solve these equations numerically?**

   The numerical scheme will depend on the choice of the vortex elements, the deformation equations, the required accuracy and computational speed, etc. Therefore, no general recommandations can be given on this issue.

## 7.2 Vortex Method Requirements

A *reasonable* [5] vortex method as to meet several requirements. Some, in our view indisputable, requirements are given in this section; their order conveys our opinion about their relative importance [6].

[4]See [121] for a review. The cut-off technique was already applied in Thomson's 1883 *Treatise* [205, §11.1].

[5]It may be clear that discussion of the meaning of *reasonable* vortex methods is difficult. It strongly depends on the intention one has in mind applying a specific vortex method. Unfortunately, most authors working with vortex methods avoid a discussion of their intentions and of the applicability of their methods.

[6]One might argue that an additional, important, requirement should be added, i.e. no undesirable effects should be caused by the numerical scheme. One example of these effects is the influence of numerical or artificial viscosity. However, Beale & Majda [22] and Engquist & Hou in [9] have stated that vortex methods contain no inherent errors which act like the numerical viscosity of conventional Eulerian difference methods. Other requirements, like correct timestep adaption, are regarded as trivial and not typically related to vortex methods.

1. **divergence-free vorticity field**

   In an inviscid flow, the net flux of vorticity normal to the surface of any closed volume $V$, given by by $\int_{\partial V} \boldsymbol{w} \cdot \boldsymbol{n}$, should be zero. This requirement can be translated into: the vorticity field should be divergence-free, i.e.

   $$\boldsymbol{\nabla} \cdot \boldsymbol{w} = 0 \tag{7.2}$$

   and is directly satisfied by a flow in which the relation (1.1) is preserved. Since this relation is the foundation of vorticity theory, this requirement is of significant importance.

2. **correct modelling of continuous distributions of vorticity**

   The vortex elements chosen for a vortex method have to provide a sufficiently accurate representation of continuous structures, like vortex rings. In general, one would like a representation which shows as many similarities in physical behaviour as possible compared with their "real" counterparts. However, this may not always be necessary or possible [7].

3. **correct representation of deformation and interaction of vortex structures**

   A correct modelling of vortex structures is no guarantee that the deformation (e.g. stretching, core deformation, stability) of vortex structures and the interaction of two or more of them (e.g. reconnection, alignment) will be correctly simulated. Comparison with experimental results is necessary to decide on this issue.

4. **conservation of motion-invariants**

   For inviscid vortex methods, the conservation of so-called motion-invariants should be satisfied. For 3-D vortex flows, the relevant invariants are total vorticity, total linear momentum, total angular momentum, total kinetic energy, and total helicity.

   Some authors (see e.g. [206]) have stated that this requirement can be reformulated as: the vorticity representation should be a weak solution of the equation describing vortex deformation. E.g., a weak solution of the Helmholtz equation has to satisfy the condition:

   $$\int f(\boldsymbol{x}) \, \{\frac{D\boldsymbol{w}}{Dt} - (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v}\} = \boldsymbol{0} \tag{7.3}$$

   for any so-called (smooth) test-function $f(\boldsymbol{x})$.

   A related requirement is conservation of the circulation ( Kelvin's Circulation Theorem; see §4.1) of any closed vortex structure, e.g. a vortex ring.

5. **no negative effects of remeshing**

   Vortex methods may allow regions with fine-scale structures to develop in an intermittent manner. Nevertheless, remeshing may be required when lack of resolution arises. Care should be taken that the remeshing scheme (changing the time and/or space step in numerical simulations) involved in any vortex method should not introduce undesirable effects, which have no physical meaning or are forbidden from a physical point of view.

[7] In the Epilogue, some additional remarks on modelling can be found.

6. **correct boundary conditions**

   Boundary conditions should be imposed correctly. For vortex methods, free-slip conditions (i.e. zero velocity normal to the wall) may be imposed relatively easily by means of "mirrored" vorticity indicated by $\boldsymbol{w}^*$; see fig.7.1. However, the application of the

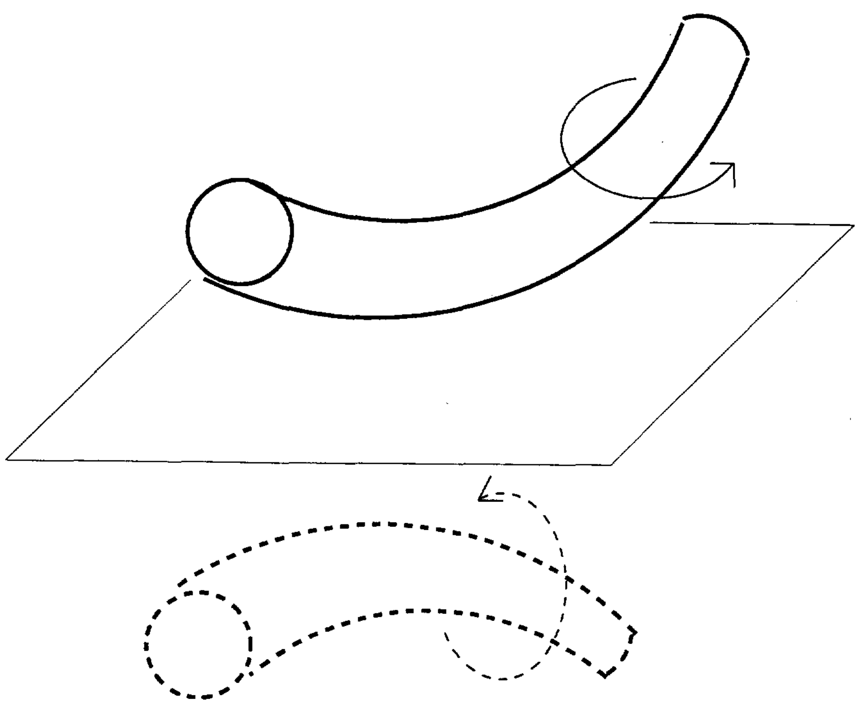

Figure 7.1: Imposing a free-slip boundary condition by addition of "mirrored" vorticity.

   no-slip condition (i.e. zero tangential velocity at the wall as required in viscous flows) usually requires much more care and effort.

7. **convergence**

   The requirement of convergence is related to the question: does the approximate solution obtained from the vortex method approach the exact solution? Convergence has been investigated both analytically and numerically for several vortex methods [8].

8. **acceptable computational effort**

   Though one of the attractions of vortex methods is the fact that computational points (vortex elements) are only required in rotational parts of the flow, the computation time is roughly proportional to the square of the number of vortex elements or coordinates. A Lagrangian method takes much computation time (usually claimed to be proportional to $N^2$, where $N$ is the number of vortex elements). On the other hand, in an Eulerian approach, also parts of the flow field have to be regarded which may be insignificant.

[8]See Caflisch in [32] for a short survey of the issue of convergence. Anderson & Greengard [8] have pointed out that an investigation of the convergence of the numerical method used in the application of a vortex method should also be investigated. A more detailed discussion of convergence in case of vortex methods is postponed to §11.1.7.

## 7.3 A Short Survey of Vortex Methods

It is not easy to provide a fairly clear, short, and general survey of the vortex methods, which have been introduced and applied in the last few decades. The conceptual questions on vortex methods treated in §7.1 allow the division into several classes, though alternatives to our method of categorization may be proposed. Below, we will give a short survey of some inviscid, mainly 3-D, vortex methods. This survey is only intended to give an impression of the "state of the art".

### 7.3.1 Vortex-Filament Methods

In these methods, the elements are vortex filaments represented by 3-D curves, on which points are defined. In the so-called thin-filament method the core of the filaments is supposed to remain nearly constant in time and the wavelengths of disturbances along the filament are supposed to be much larger than the core radius. This vortex-filament method converges without smoothing, as is claimed in [90].

However, both assumptions mentioned above are not sufficient for accurate approximations of vortex motion [121] [9]. Therefore, smoothed-vortex-filament methods have been proposed. However, a parameter has to be chosen for these methods to get an appropriately averaged velocity of a collection of vortex lines. This leads to different schemes and to a dependence of conservation of e.g. energy on this parameter [121, p. 539].

In vortex-filament methods, the rule of Biot-Savart is applied to calculate the evolution of the space curves. No account is taken of the deformation of the vorticity field.

### 7.3.2 Vortex-in-Cell Methods

A rather different method is the vortex-in-cell method. Although the vorticity field is treated in a Lagrangian way, the Poisson equation for the streamfunction is solved on an Eulerian mesh to obtain the velocity field. It appears that because of the errors involved in this method, the simulations are sensitive to the size of the mesh, the number of vortices, the time-step, etc. Besides, they require relatively much computational effort. Artificial viscosity is introduced by the numerical scheme which makes this actually a viscous vortex method. For details we refer to e.g. [122, §4] and [207, §2.6].

### 7.3.3 Vortex-Point Methods

The oldest and perhaps best-known example of the vortex-point methods is the 2-D Lagrangian unsmoothed vortex-point method, which is better known as the point-vortex method (mentioned in the Interlude; see also [122, §2] and [153]). The (scalar) vorticity field is approximated by 2-D delta-functions:

$$w = \sum_{\alpha} \Gamma_\alpha \delta(\boldsymbol{x} - \boldsymbol{x}_\alpha) \tag{7.4}$$

where $\alpha$ is the label of a point-vortex, $\Gamma_\alpha$ its strength, and $\boldsymbol{x}_\alpha$ its location. Since vortex deformation is zero for 2-D vortices, $\Gamma_\alpha$ is constant. The equations for these point-vortices can be written as a Hamiltonian system, in which the Hamiltonian is the total kinetic energy minus the so-called self-energy of the point-vortices. This subtraction of self-energy is necessary due to its infinity caused by the singularities in the velocity field at the locations of the point-vortices [10].

[9] These same restrictions had hindered Kelvin in his investigation of the vibrations of a "columnar vortex" [250]; see §5.3.4.

[10] Campbell in [32] has applied a vortex lattice to simulate point-vortex dynamics. In this method one also encounters the "problem" of self-energy of the vortices. However, he claims that self-energy does not affect the

Only recently the convergence of the point-vortex method for the 2-D incompressible Euler equation with smooth solutions has been proved (see e.g. [140]).

For 3-D flow simulations several vortex-point methods have been introduced.

One example of unsmoothed point-vortex methods is the one presented by Hou & Lowengrub [90]. In their set-up, vorticity is defined on a grid and a grid size is introduced. The displacement of the grid points is calculated from the rule of Biot-Savart. The Biot-Savart kernel is not regularized, since the authors show that for this method the kernel has a "natural cut-off" or vortex core. Vortex deformation is calculated by means of the Cauchy vorticity formula. Hou & Lowengrub claim to have proved the stability and convergence of this method.

Anderson & Greengard [8] have introduced two smoothed versions of the method of Hou & Lowengrub: one in which the Cauchy vorticity formula is retained and another one in which this formula has been replaced by the Helmholtz equation. For the former they have proved convergence. A detailed mathematical treatment of this method can be found in [70].

The grid-less variant of the Hou & Lowengrub method is the so-called vorton method as introduced by Novikov [168]. This 3-D method can be regarded as being closest to the 2-D point-vortex method. The essence of this method will be treated in Chapter 8, where we introduce our own (improved) vorton method. The second smoothed version as introduced by Anderson & Greengard has become known as the soft-vorton method, the smoothed version of Novikov's vorton method [11].

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dynamics.

[11]The soft-vorton method is briefly discussed in Appendix B.

Chapter 8

# The Vorton Method

## 8.1 Introduction

For the vortex methods mentioned in §7.3, several drawbacks have been mentioned. The most important ones will be repeated here.

For the use of any vortex element other than the unsmoothed vortex point, a drawback with regard to computational effort exists. Vortex methods involving vortex blobs (i.e. smoothed vortex-points) and (smoothed) vortex filaments often require a large amount of detailed information for the tracking of their location, strength, *and* vorticity distribution. Therefore, vortex points are preferable since they require less book-keeping.

With regard to the use of smoothed vortex elements, whether vortex points or vortex filaments, the somewhat arbitrary choice of the smoothing function has to be mentioned. It has become clear from literature that the numerical results also depend on this choice. Another characteristic of vortex methods using smoothed vortex elements is the requirement of overlap of the vortex elements, i.e. the distance between vortex elements has to be smaller than the sum of the characteristic dimensions of the smoothed vorticity distributions around them. According to Sarpkaya, in the overlap regions Helmholtz's First Theorem is not valid anymore and conservation of energy is violated [207, p.11].

Another important objection to all vortex-point methods mentioned in §7.3.3, is the fact that the vorticity field is not divergence-free in general (requirement 1 in the list of §7.2).

The vorton method is an unsmoothed vortex-point method according to the terminology introduced in §7.3. In this chapter the set-up of this method will be presented. Its original version, first elaborated by Novikov [1] [168] in 1983 and to be called the "original vorton

[1]Novikov has mentioned that his inspiration has partly come from the theory of superfluid vortices. Superfluidity has appeared to be one of the most successful applications of vorticity theory outside the field of classical fluid mechanics. In superfluid helium, a form of liquid helium that flows without viscosity or friction, very thin vortex filaments have been found experimentally with cores of atomic order of magnitude [48]. Researchers in this area have developed numerical simulation methods which show much resemblance to the vortex-filament method (see §7.3.1). Reconnection of these quantized vortices happens when distances between vortex lines are on atomic scales and thus involve quantum mechanics; it is claimed that on this scale there is no violation of Kelvin's Circulation Theorem (this suggests that also in inviscid flows reconnection might take place). However, in certain respects superfluid vortex motion is different from that in ordinary fluids. Vortex stretching does not take place, due to the quantization of circulation. Furthermore, at a given temperature and pressure a quantized vortex must always have the same core radius. The last few years, numerical studies of tangles of superfluid vortices have profited from the development of vortex methods in hydrodynamics [1].

Another vortex structure playing an important role in superfluids is the roton, a quantum-mechanical microscopic vortex ring. Brush [30] remarked, after a short discussion of Kelvin's work on vortex rings:

> ... a succesful investigation of the interaction of two vortex rings, in their modern reincarnation as rotons, would be an important contribution toward the synthesis of quantum mechanics and hydrodynamics, and toward the construction of a theory which promises to achieve a consistent and unambiguous deduction of observable properties from postulates about collective molecular motions, without attempting the apparently hopeless task of describing the motions of all the

method", may be regarded as the 3-D counterpart of 2-D point-vortex method (see §7.3.3), since it is based on a vorticity field consisting of 3-D delta-functions (compare (7.4)). These have been called "vortons" [2]. However, in 3-D flows we have to take into account the deformation of the vorticity field; this phenomenon is absent in 2-D flows.

Though the equations derived by Novikov for the displacement and deformation of vortons (the **vorton equations**) have appeared to provide the basis for seemingly correct simulations of vortex motion, it has also been realized that his derivation suffers from the fact that the vorticity field he applied is not divergence-free. This same weakness in Novikov's derivation causes the inconsistency between his vorton deformation equation and that derived by Kuwabara (see e.g. [114]). The only difference between these derivations is the choice of the basic equation: for Novikov this has been the Helmholtz equation (2.2), for Kuwabara it has been the so-called transposed representation of the Helmholtz equation.

To clarify the different representations of the Helmholtz equation and for convenience in our further elaboration in §8.3, we rewrite (2.2) as:

$$\frac{D\boldsymbol{w}}{Dt} = (\boldsymbol{v}') \circ \boldsymbol{w} \tag{8.1}$$

where the matrix $(\boldsymbol{v}')$ is defined by:

$$\begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} & \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} & \frac{\partial v_2}{\partial z} \\ \frac{\partial v_3}{\partial x} & \frac{\partial v_3}{\partial y} & \frac{\partial v_3}{\partial z} \end{pmatrix} \tag{8.2}$$

where $v_1, v_2, v_3$ are the components of vector $\boldsymbol{v}$.

It can be shown that for vorticity fields satisfying relation (1.1) the Helmholtz equation (2.2), which in Cartesian coordinates reads:

$$\frac{Dw_i}{Dt} = \frac{\partial v_i}{\partial x_j} w_j \ ,$$

can be reformulated into the **transposed** representation of the Helmholtz equation:

$$\frac{Dw_i}{Dt} = \frac{\partial v_j}{\partial x_i} w_j$$

individual particles. The possibility of constructing such theories is of considerable significance for both physics and chemistry. [30, p.536]

Whereas, as discussed in §5.3.2, a fundamental critiscism of the vortex atom concerned its decrease of velocity with increasing energy, for the roton this is just the desired property!

Also in the related field of superconductivity, the vortex concept has gained a much-studied position. Recently, vortex analogies have also been found in lasers (see Weiss *et al.* in [33]).

For a popular account of superfluids, we refer to [48]; a text-book on both superfluidity and superconductivity is [263].

[2]The term "vorton" has been proposed in other areas of physics. In particle physics the vorton is defined as a stationary vortex ring, which can spin, have electric charge and behaves like a magnetic dipole. In many ways they are like ordinary particles, hence the name vorton in analogy with electron, photon, etc. Related to this quantum-scale vorton and based on its properties, "cosmic vortons" have been proposed. Generally, it is suggested that vortices, or cosmic strings, appear in a cosmological phase transition. During the period of the early universe, vorton-like structures which may even have extended over astronomical distances, could have been formed out of a Brownian network of vortices [45]. The name "vorton" has also been given to a monopole configuration with electromagnetic charge whose fields satisfy Maxwell's equations [61]. If such vortons should really exist as physical particles, "they would be quite different from presently known particles".

or

$$\frac{D\boldsymbol{w}}{Dt} = (\boldsymbol{v}')^* \circ \boldsymbol{w} \tag{8.3}$$

where $(\boldsymbol{v}')^*$ is the transposed of $(\boldsymbol{v}')$.

Consequently, a third formulation may be the **mixed** representation:

$$\frac{D\boldsymbol{w}}{Dt} = \frac{1}{2}[(\boldsymbol{v}') + (\boldsymbol{v}')^*] \circ \boldsymbol{w}\,. \tag{8.4}$$

Our derivation of the vorton deformation equation differs from Novikov's approach in two ways. First, it is based on a divergence-free vorticity field, which is an extension of Novikov's field; this is the subject of §8.2. Second, in §8.3 we apply a so-called weak formulation of the continuous equations from which the vorton equations are derived. This leads to a new vorton deformation equation and a proof that the inconsistency between Novikov's and Kuwabara's equations can only be removed by the use of a divergence-free vorticity field [3].

## 8.2 The Vorton Fields

In the original vorton method, the vorticity field is represented by:

$$\boldsymbol{w}(\boldsymbol{x},t) = \sum_{\alpha} \boldsymbol{\gamma}_\alpha(t)\; \delta(\boldsymbol{R}_\alpha(\boldsymbol{x},t)) \tag{8.5}$$

where $\boldsymbol{R}_\alpha(\boldsymbol{x},t) \equiv \boldsymbol{x} - \boldsymbol{r}_\alpha(t)$, $\delta(...)$ is the 3-D delta-function, and summation is over all vortons. The vortons are determined by a label $\alpha$, a location vector $\boldsymbol{r}_\alpha$, and a strength vector $\boldsymbol{\gamma}_\alpha$ (which depends on time only).

The vector $\boldsymbol{\gamma}_\alpha$ has dimension of (volume/time) and can be regarded as a local vorticity distribution, given by:

$$\boldsymbol{\gamma}_\alpha \sim \int_{V_\alpha} \boldsymbol{w} \tag{8.6}$$

where the volume $V_\alpha$ around the location $\boldsymbol{r}_\alpha$ of vorton $\alpha$ gets infinitesimally small, while the integral remains finite.

From (8.5) Novikov derived the velocity field by applying the rule of Biot-Savart (2.3), giving:

$$\boldsymbol{v}(\boldsymbol{x},t) = \frac{1}{4\pi} \sum_{\alpha} \frac{\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_\alpha}{R_\alpha^3} \tag{8.7}$$

where $R_\alpha \equiv |\,\boldsymbol{R}_\alpha\,|$.

However, the field (8.5) is not divergence-free. Consequently, it does not satisfy one of the basic requirements of vortex methods (see §7.2) and consequently the rule of Biot-Savart may not be applied. In order to derive a divergence-free vorticity field, we have to take account of definition (1.1). The following procedure takes account of this requirement, while at the same time it provides a divergence-free velocity field and avoids application of the rule of Biot-Savart:

$$\text{vector potential } \boldsymbol{A} \;\overset{\boldsymbol{v}=\nabla\times\boldsymbol{A}}{\Longrightarrow}\; \text{velocity } \boldsymbol{v} \;\overset{\boldsymbol{w}=\nabla\times\boldsymbol{v}}{\Longrightarrow}\; \text{vorticity } \boldsymbol{w}\,.$$

[3] An alternative to the (original) vorton method is the soft-vorton method (already mentioned in §7.3.3) which is discussed in Appendix B.

For the vorton vector-potential field we take [4]:

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x},t) &= \frac{1}{4\pi}\sum_\alpha \frac{\boldsymbol{\gamma}_\alpha}{R_\alpha} \\ &= \sum_\alpha \phi(\boldsymbol{R}_\alpha)\,\boldsymbol{\gamma}_\alpha \end{aligned} \tag{8.8}$$

where we have introduced the function:

$$\phi(\boldsymbol{x}) \equiv \frac{1}{4\pi x} . \tag{8.9}$$

In mathematical terms $\phi$ is a Green's function, which is defined as the solution of the Poisson equation

$$\nabla^2\phi = -\delta(\boldsymbol{x}) .$$

From this field we find the following velocity field:

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x},t) &= \sum_\alpha \boldsymbol{\nabla}\phi(\boldsymbol{R}_\alpha)\times\boldsymbol{\gamma}_\alpha \\ &= \frac{1}{4\pi}\sum_\alpha \frac{\boldsymbol{\gamma}_\alpha\times\boldsymbol{R}_\alpha}{R_\alpha^3} \end{aligned} \tag{8.10}$$

where $\boldsymbol{\nabla}\phi(\boldsymbol{R}_\alpha) \equiv (\boldsymbol{\nabla}\phi)|_{\boldsymbol{x}=\boldsymbol{R}_\alpha}$.

Finally, from (8.10) we get the following vorticity field by applying (1.1) [5]:

$$\begin{aligned} \boldsymbol{w}(\boldsymbol{x},t) &= \sum_\alpha \{\boldsymbol{\gamma}_\alpha\,\delta(\boldsymbol{R}_\alpha) + \phi''(\boldsymbol{R}_\alpha)\circ\boldsymbol{\gamma}_\alpha\} \\ &= \sum_\alpha \{\boldsymbol{\gamma}_\alpha\,\delta(\boldsymbol{R}_\alpha) + \boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot(\phi(\boldsymbol{R}_\alpha)\boldsymbol{\gamma}_\alpha)]\} \\ &= \sum_\alpha \{\boldsymbol{\gamma}_\alpha\,\delta(\boldsymbol{R}_\alpha) - \frac{1}{4\pi}[\frac{\boldsymbol{\gamma}_\alpha}{R_\alpha^3} - \frac{3\boldsymbol{R}_\alpha(\boldsymbol{R}_\alpha\cdot\boldsymbol{\gamma}_\alpha)}{R_\alpha^5}]\} \end{aligned} \tag{8.11}$$

where

$$\phi''(\boldsymbol{R}_\alpha) \equiv \phi''|_{\boldsymbol{x}=\boldsymbol{R}_\alpha} \tag{8.12}$$

and the matrix $\phi''$ is defined by:

$$\begin{pmatrix} \frac{\partial^2\phi}{\partial x\partial x} & \frac{\partial^2\phi}{\partial y\partial x} & \frac{\partial^2\phi}{\partial z\partial x} \\ \frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial y\partial y} & \frac{\partial^2\phi}{\partial z\partial y} \\ \frac{\partial^2\phi}{\partial x\partial z} & \frac{\partial^2\phi}{\partial y\partial z} & \frac{\partial^2\phi}{\partial z\partial z} \end{pmatrix} . \tag{8.13}$$

Comparing this result to the original vorton field (8.5), we conclude that the second (gradient) part of (8.11), which is due to the the vector potential field not being divergence-free,

[4]The choice of the vector potential remains to be elucidated. We have chosen a function $\phi$ such that Novikov's velocity field (8.7) is obtained. If, for example, we require a divergence-free field $\boldsymbol{A}$, the choice of $\boldsymbol{A}$ (or $\phi$) is restricted as explained in Appendix A. At the end of this Appendix we further discuss our vorton vector potential.

A more general treatment of (8.8), without determining function $\phi$, can be found in Appendix B where the soft-vorton method is treated.

[5]This expression can also be found in Novikov's original paper [168].

renders the vorticity field divergence-free. It can be regarded as the nonlocal vorticity field surrounding the original singular vorton represented by the first part of (8.11). In a plane parallel to the strength vector and going through the location vector, the vortex lines of the nonlocal vorticity field resemble the "coil-like" streamline pattern of a doublet or dipole; see fig.8.1.

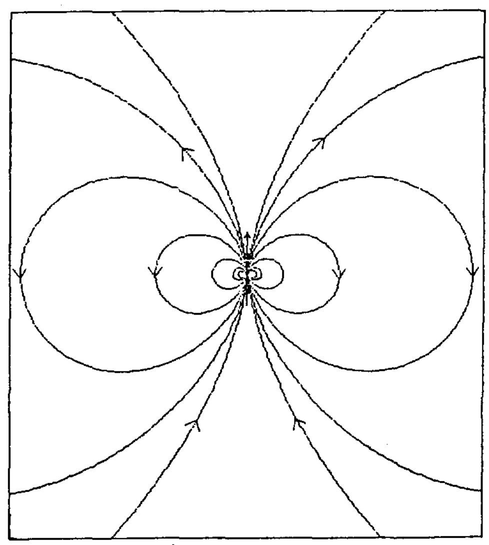


Figure 8.1: Vortex lines around one vorton (located at center, directed upwards).

Below, it will appear that this nonlocal part is crucial for a consistent derivation of the vorton deformation equation. Though it does not contribute to the integral of the rule of Biot-Savart (2.3), i.e. to the velocity field, it is of fundamental importance to a correct derivation of the vorton deformation equation, as will be shown below.

Mathematically speaking, the original vorticity field (8.5) has been projected onto a divergence-free field (8.11), without affecting the convolution with the Biot-Savart kernel.

## 8.3 The Vorton Equations

As indicated before, vorton dynamics consists of vorton displacement and vorton deformation. Below, the equations which describe both phenomena are presented; details of the derivation can be found in Appendix C. For the deformation equation, its superiority over Novikov's and Kuwabara's equations is discussed and an interpretation of our equation is attempted.

Derivation of the vorton displacement euqation is elementary. It follows directly from relation (7.1), which can be formulated here as:

$$\dot{\boldsymbol{r}}_\alpha = \tilde{\boldsymbol{v}}^\alpha(\boldsymbol{r}_\alpha, t) \tag{8.14}$$

where $\dot{\boldsymbol{r}}_\alpha \equiv d\boldsymbol{r}_\alpha/dt$ and the tilde indicates the field induced by all vortons except $\alpha$ [6].

[6]This suggests that "self-displacement" of vortons is eliminated. Justification for this elimination is given in the derivation presented in Appendix C.

Elaboration of (8.14), by applying (8.10), gives:

$$\dot{\boldsymbol{r}}_\alpha = \sum_{\beta\neq\alpha} \boldsymbol{\nabla}\phi(\boldsymbol{R}_{\alpha\beta}) \times \boldsymbol{\gamma}_\beta \tag{8.15}$$

where the function $\phi$ is given by (8.9) and $\boldsymbol{R}_{\alpha\beta} \equiv \boldsymbol{r}_\alpha - \boldsymbol{r}_\beta$ .

With regard to vorton deformation, it is essential to remove the earlier mentioned inconsistency between the vorton deformation equations derived by Novikov and Kuwabara. We have devised a method for deriving vorton deformation equations from both representations of the Helmholtz equation which do not show any inconsistency. Its full details are revealed in Appendix C; below we give a short indication and the resulting equations.

We split the vorton velocity and vorticity fields derived in §8.2 into two parts. One part is the field induced by a vorton $\alpha$ for which we want to derive the deformation equation: velocity $\boldsymbol{v}^\alpha(\boldsymbol{x},t)$ and vorticity $\boldsymbol{w}^\alpha(\boldsymbol{x},t)$. The other part is the contribution from all other vortons, which is indicated by a tilde: $\tilde{\boldsymbol{v}}^\alpha(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{w}}^\alpha(\boldsymbol{x},t)$.

The fundament of our approach is a weak formulation of the vortex deformation equations, i.e. these equations will be integrated about a small sphere $B_\alpha$ with radius $\epsilon$ and centred around the vorton location $\boldsymbol{r}_\alpha$. It is assumed that $\epsilon$ is so small that no other vorton locations are inside sphere $B_\alpha$ [7].

Using the splitting of the velocity and vorticity fields mentioned above, we get from (8.1) the following expression for the vorton deformation equation (in case of the Helmholtz equation):

$$\int_{B_\alpha} \frac{D(\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha)}{Dt} = \int_{B_\alpha} ((\boldsymbol{v}^\alpha + \tilde{\boldsymbol{v}}^\alpha)') \circ (\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha) \; . \tag{8.16}$$

Elaboration of expression (8.16) shows that the vorton deformation equation can be written as (see also (C.4) in Appendix C):

$$\dot{\boldsymbol{\gamma}}_\alpha = N_\alpha + S_\alpha \tag{8.17}$$

where

$$N_\alpha \equiv \sum_{\beta\neq\alpha} [\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\alpha] \times \boldsymbol{\gamma}_\beta$$

and

$$S_\alpha \equiv \frac{1}{2} \sum_{\beta\neq\alpha} \boldsymbol{\gamma}_\alpha \times [\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\beta] \tag{8.18}$$

where definition (8.12)has been used.

In the same way, from the transposed Helmholtz equation (8.3) one can derive (see also (C.6) in Appendix C):

$$\dot{\boldsymbol{\gamma}}_\alpha = K_\alpha - S_\alpha \tag{8.19}$$

where

$$K_\alpha \equiv \sum_{\beta\neq\alpha} \phi''(\boldsymbol{R}_{\alpha\beta}) \circ (\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha) \; .$$

Elaboration of (8.17) and (8.19) shows that both vorton deformation equations are equivalent, i.e. $N_\alpha + S_\alpha = K_\alpha - S_\alpha$.

[7] Note the resemblance of this approach to definition (7.3) of so-called weak solutions given in §7.2.

If we compare the deformation equations derived here with those found by Novikov [168] from the basic representation of the Helmholtz equation and by Kuwabara [114] from the transposed representation of the Helmholtz equation, we remark that Novikov's original vorton deformation equation is related to our expression $N_\alpha$, i.e.:

$$\dot{\boldsymbol{\gamma}}_\alpha = N_\alpha \,. \tag{8.20}$$

This equation will be called the **N-equation.**

Kuwabara's vorton deformation equation is related to our expression $K_\alpha$, i.e.:

$$\dot{\boldsymbol{\gamma}}_\alpha = K_\alpha \,. \tag{8.21}$$

This equation will be called the **K-equation.**

The inconsistency between Novikov's and Kuwabara's equations can now be stated as: $N_\alpha \neq K_\alpha$ Addition of the expression $S_\alpha$ to $N_\alpha$ and substraction of $S_\alpha$ from $K_\alpha$ causes complete equivalence. Remark that $S_\alpha$ is related to the nonlocal part of the vorton vorticity field (8.11), which explains why it is not present in the deformation equations derived by Novikov and by Kuwabara, who applied only the local field (8.5).

By adding equations (8.17) and (8.19), $S_\alpha$ disappears:

$$\dot{\boldsymbol{\gamma}}_\alpha = \frac{1}{2}\{N_\alpha + K_\alpha\} \,. \tag{8.22}$$

This will be called the **N+K-equation.**

Equation (8.22) suggests that $S_\alpha$ does not contribute to stretching of the vortons. This can be seen by taking the vector product of $S_\alpha$ (given by (8.18)) and $\boldsymbol{\gamma}_\alpha$.

Comparison of the expression (8.18) with the Euler equation for rigid body motion shows that $S_\alpha$ can be interpreted as a pure "spin" contribution. This can be demonstrated by regarding the rotation of a rigid body. For the angular momentum $\boldsymbol{M}$ of a rigid body we have:

$$\boldsymbol{M} = \boldsymbol{I} \circ \boldsymbol{\Omega}$$

where $\boldsymbol{I}$ is its inertia tensor and $\boldsymbol{\Omega}$ its angular velocity. Furthermore, for the rate of change of the angular momentum in time, we have:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega} \,.$$

Combining these equations, we get:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times (\boldsymbol{I}^{-1} \circ \boldsymbol{M}) \,.$$

Compare this expression with the vorton deformation due to $S_\alpha$, i.e.

$$\dot{\boldsymbol{\gamma}}_\alpha = S_\alpha = \sum_{\beta \neq \alpha} \boldsymbol{\gamma}_\alpha \times [\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\beta)] \,.$$

Some insight into expression (8.22) can be gained by observing that it can be rewritten as:

$$\dot{\boldsymbol{\gamma}}_\alpha = \frac{\partial G_\alpha}{\partial \boldsymbol{\gamma}_\alpha} \tag{8.23}$$

where

$$G_\alpha \equiv \frac{1}{2}\sum_{\beta\neq\alpha}[\phi''(\boldsymbol{R}_{\alpha\beta})\circ\boldsymbol{\gamma}_\alpha]\cdot(\boldsymbol{\gamma}_\beta\times\boldsymbol{\gamma}_\alpha)\,. \tag{8.24}$$

In its turn, expression (8.24) may be rewritten as:

$$G_\alpha = \frac{\partial H_\alpha}{\partial \boldsymbol{r}_\alpha}\cdot\boldsymbol{\gamma}_\alpha \tag{8.25}$$

where

$$H_\alpha \equiv \frac{1}{2}\sum_{\beta\neq\alpha}\boldsymbol{\gamma}_\alpha\cdot\boldsymbol{v}^\beta(\boldsymbol{r}_\alpha)\,.$$

$H_\alpha$ may be called the local helicity density, defined as the integrand of total helicity (see §C of the Interlude). Apparently, the deformation of vorton $\alpha$ is related to the change of helicity density at its location caused by all vortons $\beta \neq \alpha$, due to the movement of vorton $\alpha$ in the direction of its strength vector $\boldsymbol{\gamma}_\alpha$.

From (8.23) and (8.25) we derive:

$$\dot{\boldsymbol{\gamma}}_\alpha = \frac{\partial}{\partial\boldsymbol{\gamma}_\alpha}\{\frac{\partial H_\alpha}{\partial \boldsymbol{r}_\alpha}\cdot\boldsymbol{\gamma}_\alpha\}$$

from which follows:

$$\dot{\boldsymbol{\gamma}}_\alpha = (\frac{\partial}{\partial\boldsymbol{\gamma}_\alpha}\{\frac{\partial H_\alpha}{\partial \boldsymbol{r}_\alpha}\})^*\circ\boldsymbol{\gamma}_\alpha + \frac{\partial H_\alpha}{\partial \boldsymbol{r}_\alpha}\,. \tag{8.26}$$

Comparison with (8.22) shows the first part of (8.26) to be equal to $\frac{1}{2}N_\alpha$ while the second part is equal to $\frac{1}{2}K_\alpha$ [8].

Since $G_\alpha$ (8.24) is homogeneous of second degree in $\boldsymbol{\gamma}_\alpha$, we find the following expression for the stretching of vorton $\alpha$:

$$\frac{d\gamma_\alpha^2}{dt} = 4G_\alpha\,.$$

We conclude that levels of constant stretching of vorton $\alpha$ are equal to levels of constant $G_\alpha$.

To investigate these levels of constant nonzero vorton stretching, we regard the configuration of two vortons, 1 and 2. Without violating generality, we take

$$\boldsymbol{r}_1 = \begin{pmatrix}0\\0\\0\end{pmatrix},\ \boldsymbol{\gamma}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix},\ \boldsymbol{r}_2 = \begin{pmatrix}x\\y\\z\end{pmatrix},\ \boldsymbol{\gamma}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

We investigate levels on which vorton 2 has constant stretching due to vorton 1. From (8.24) we find surfaces given by $x\,y = C\,(x^2+y^2+z^2)^{5/2}$, where $C$ is a constant. In fig.8.2, the surface for $C = 1$ has been drawn. We see that the area of influence of vorton 1 is symmetrical around its strength vector. The four "lobes" are separated by the zero-stretching surfaces $G_2 = 0$, which in this case are given by $x = 0$ and $y = 0$. Two oppositely placed lobs have the same stretching levels, of either positive or negative value.

[8] This last result has also been derived by Kuwabara (see e.g. [114]).

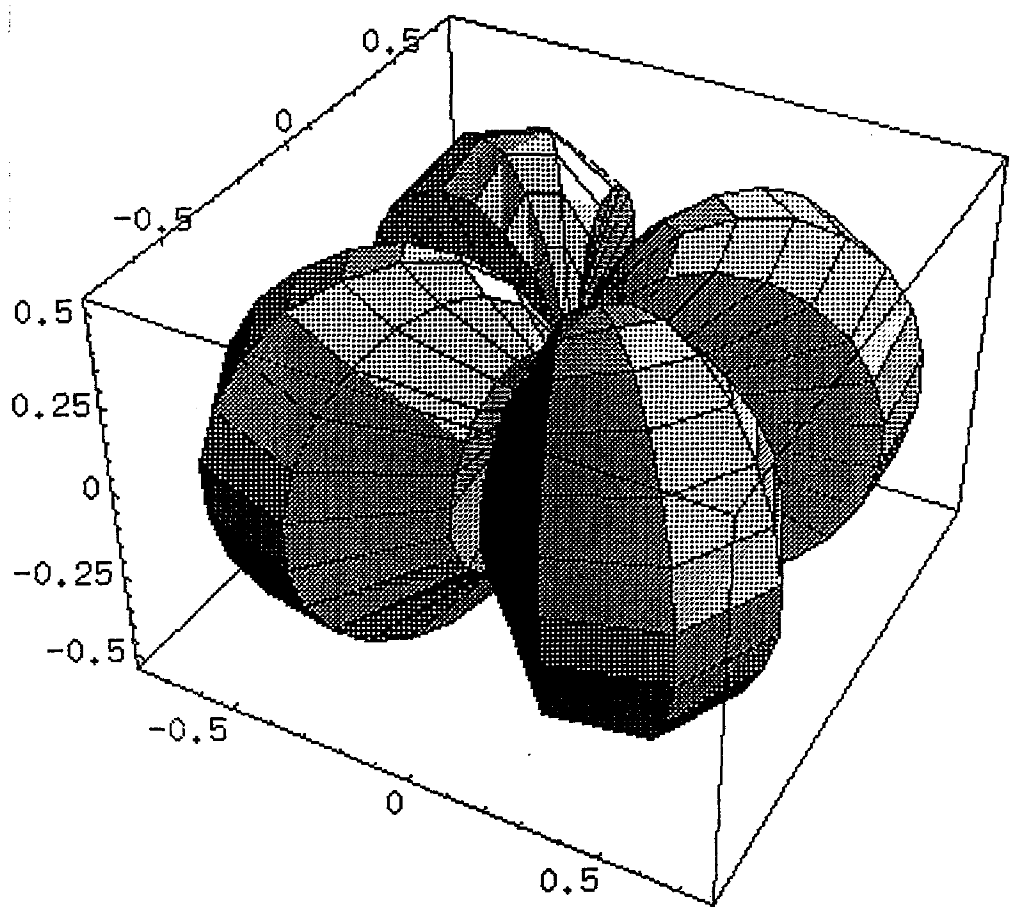



Figure 8.2: Surface given by $xy = (x^2 + y^2 + z^2)^{5/2}$, illustrating the influence of vorton 1 on vorton 2 (see text).

# Chapter 9

# Numerical Simulations: Preparatory Remarks

The vorton equations, presented in §8.3, will be used to simulate the dynamics of continuous vortex structures. This is done by constructing a discrete vorton representation of these structures and solving the vorton equations by means of a numerical scheme. Comparison of the numerical results with results from other numerical simulations and from experiments will clarify the applicability of the vorton method. More specifically, the simulation will provide information on the fulfilment of requirements 2 (correct modelling of continuous distributions of vorticity), 3 (correct representation of deformation and interaction of vortex structures), 4 (conservation of motion-invariants), and 5 (no negative effects of remeshing) as posed in §7.2. If we find a satisfactory performance of our vorton method and agreement with results found by others, confidence in the applicability of the method to other, more complicated, flow phenomena, will be established.

In the last decade, several vorton simulations have been reported in literature in which both the N-equation (8.20) and the K-equation (8.21) have been used. We mention those by Novikov in [168] and [170]; by Novikov *et al.* in [6]; by Novikov and Aksman in [5]; and by Kuwabara in e.g. [113] and [114].

Besides, at least two dissertations have been (partly) devoted to the vorton method and its application to numerical simulations. Pedrizzetti [178] applied the N-equation [1], while Winckelmans [283] studied both the N-, K-, and N+K-equations and also regarded the soft-vorton method (see Appendix B) and the possibility of adding viscous diffusion to the vorton equations [2].

In Chapter 10, we will present the results of the numerical simulations of several test cases. In this preparatory chapter, we discuss the choice of these cases and their relation to the aim of our research (§9.1); we give some details of the numerics (§9.2); we introduce the so-called diagnostics which are used in our evaluation of the numerical results (§9.3); and we describe the technique of vorton division whose applicability and value will be investigated by means of some of the simulations of Chapter 10 (§9.4).

## 9.1 Aims and Choices

The choice of the test cases which have been regarded and the choice of the characteristics and diagnostics calculated for these cases have been directed by the following considerations (compare some of the requirements mentioned in §7.2):

[1] Note that Pedrizzetti has added a "divergence filtering procedure". Apparently, this seemed to him the only way to ascertain reasonable results from his simulations, despite the non-divergence-free vorticity field. This procedure tends to align the vorton strength vectors $\boldsymbol{\gamma}$ with the local vector $\nabla \times \boldsymbol{v}$. However, he admitted that the filtering has "no clear physical meaning" [177].

[2] Winckelmans has applied a technique called "relaxation of the vorticity divergence" to ensure a (almost) divergence-free vorticity field. This technique requires the solving of a system of linear equations. However, it appears that these equations act dissipatively.

- **availability of experimental, numerical, or analytical data**

  Only when comparison of the simulation results with the same kind of data obtained by others is possible, an evaluation of our vorton method (i.e. application of the N+K-equation) and its applicability can be made. We realize that experimental data always relate to viscous vortex structures, and that the influence of viscosity may not always be disregarded.

- **possibility of the representation of elementary vortex phenomena**

  Among the many phenomena related to vortex motion, we have tried to find test cases which, in a generic way, allow the study of vorticity deformation (stretching), vortex core deformation, vortex reconnection, and alignment of anti-parallel vortex tubes (see §C of the Interlude).

- **possibility of the imposition of boundary conditions**

  In case of the vorton method the possibilities to impose boundary conditions are limited. As has already been indicated by Novikov [169], only the free-slip boundary condition can be realized. This can be done by adding so-called mirrored vortons, similar to the situation sketched in fig.7.1. For the simulation of non-closed infinite vortex filaments periodic boundary conditions should be imposed, requiring a substantial extra amount of computational effort. We have restricted the simulations primarily to (elliptical) vortex ring in 3-D unbounded space or near a planar free-slip boundary.

The two main aims of our numerical research have been to:

- investigate the characteristics of the vorton method introduced in Chapter 8 (i.e. the N+K-equation (8.22) for the vorton deformation): its possible applications, its limitations with regard to simulation of flow phenomena, and its satisfaction of the vortex method requirements as mentioned in §7.2;
- compare the simulations obtained from applying both the N-, the K-, and the N+K-equation; though we have shown in §8.3 by means of theoretical arguments that the last one is to be preferred, we will find confirmation of this claim by means of the numerical results.

All test cases chosen for our numerical simulations involve the interaction of vortex rings. Several reasons exist to concentrate on this vortex structure:

- As indicated in the vortex-atom-part and the Interlude, the vortex ring has been the subject of long-standing research into several of its aspects. Today it is still one of the most studied vortex structures.
- Furthermore, a vortex ring may be characterized by only a few quantities, i.e. the ring radius $R$, the number of vortons $N$, and the circulation $\Gamma$; see fig.2.3 [3].
- A vorton representation of the vortex ring is relatively easy: the vorton locations $\boldsymbol{r}_\alpha$ are put at equal distances on a circle of radius $R$, and the vorton strength vectors $\boldsymbol{\gamma}_\alpha$ - all of the same modulus $\gamma$ - are tangential to this circle; see fig.9.1 [4].

[3]However, as the reader may already realize, real vortex rings are also characterized by a distribution of vorticity in the ring's core.

[4]Other representations of vortex rings by means of vortons can be imagined. One example is shown in fig.10.28 in §10.4.2. Alternatives like these will not be investigated in this thesis, but we will return to their

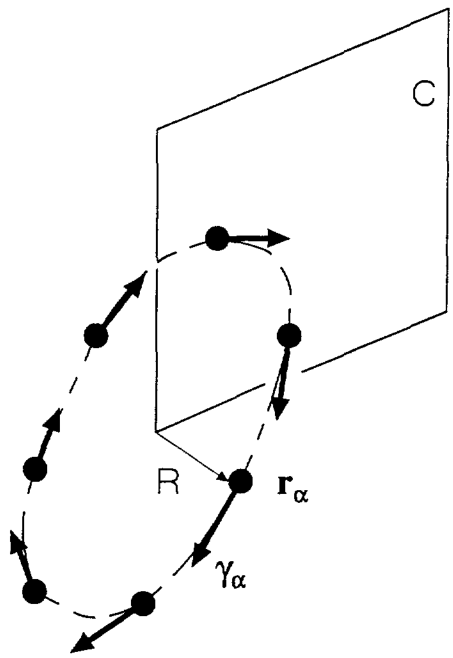



Figure 9.1: A vorton ring; vortons are indicated by arrows. (Curve $C$ is related to calculation of the ring's circulation.)

- Despite their simple appearance, the interaction of vortex rings is far from simple and trivial and shows the elementary vortex phenomena which we want to study (see above).
- Vortex rings are closed structures, requiring no special attention with regard to boundary conditions in the numerical simulations as in case of (infinite) vortex filaments (see above).
- Finally, vortex rings have been proposed as candidates for the role of coherent structures in turbulence (see §B of the Interlude and §10.6). Therefore, knowledge on their behaviour may increase our understanding of turbulent phenomena.

The following six test cases have been chosen, the essence of which will shortly be explained here:

- **single vorton ring**

  Before regarding the interaction of several circular vorton rings or the behaviour of a single noncircular vorton "ring", one should get an impression of the characteristics (e.g. its core) and the behaviour of a single circular vorton ring (e.g. stability). Since a single circular vortex ring will not exhibit deformation without influences from other vortex structures or boundaries, the simulation results for this case will not depend on the vorton equations applied and consequently no comparison has been made between the three different vorton equations.

possible significance in Chapter 11.

- **single pseudo-elliptical vortex ring**

  The simplest variation on the circular vorton ring is an elliptically shaped "ring". For reasons of convenience we only regard the so-called pseudo-elliptical vorton ring (see fig.10.7). Contrary to the circular ring, an elliptical ring will deform during translation; therefore, it forms a test case for the three different vorton equations. It turns out that the behaviour of this ring depends on the aspect ratio of the ellipse and can involve reconnection. Therefore, it is a good test case for the simulation of this last phenomenon.

- **head-on collision of two coaxial vorton rings**

  This configuration consists of two identical circular vortex rings which approach each other along a common axis. It is especially attractive since it has been treated analytically (though only for Kelvin-rings; see §A.2 of the Interlude). Besides, it is suited for the study of core deformation during close approach of vortex tubes.

- **oblique interaction of two initially parallel vortex rings**

  This configuration is one of the most elementary test cases showing the phenomenon of vortex reconnection. Besides, it is one of the few which can be compared rather extensively with experimental results.

- **interaction of two knotted vortex rings**

  This configuration, though unfortunately it cannot be studied experimentally, is one of the most elementary test cases to investigate the phenomenon of vortex alignment.

- **single vorton ring in a shear flow above a flat plate**

  This configuration is chosen to increase our insight into the behaviour of coherent structures in a boundary layer. Though the model may be too simple to describe all aspects of coherent structures, at least it shows the applicability of the vorton method in this kind of research and the influence of shear flow and (free-slip) walls.

## 9.2 Details of the Numerical Simulations

In most of the simulations to be presented in the next section, a so-called **standard vorton ring** has been used. The standard vorton ring is characterized by circulation $\Gamma = 820\ cm^2/s$ and radius $R = 0.8\ cm$.

All simulations have been carried out on an HP9000 835 minicomputer. Unless otherwise stated, the N+K-equation (8.22) has been used. To solve the vorton equations, which are ordinary differential equations, use has been made of a 4th order Runge-Kutta method.

The time step $\Delta t$ has been adapted constantly during the simulation to a value given by $R_{min}^3/\gamma_{max}$: $\Delta t = CR_{min}^3/\gamma_{max}$ ($C$ is constant). $R_{min}$ is the minimum distance between any two vortons [5] and $\gamma_{max}$ is the maximum vorton strength of the two vortons between which $R_{min}$ occurs: $\gamma_{max} = \max(\gamma_1, \gamma_2)$. See fig.9.2. This procedure is based on the following consideration: if we consider vorton 2 of this pair of vortons as a passive particle, we find from the vorton displacement equation (8.15) that it will turn around vorton 1 in a time proportional to the expression mentioned above. The proportionality factor $C$ used in our simulations has been of

[5] This minimum distance $R_{min}$ only concerns the distances between vortons not belonging to the same vortex structure, i.e. not in the same vorton ring. This time step adaptation procedure has not been applied in case of the pseudo-elliptical vorton ring.

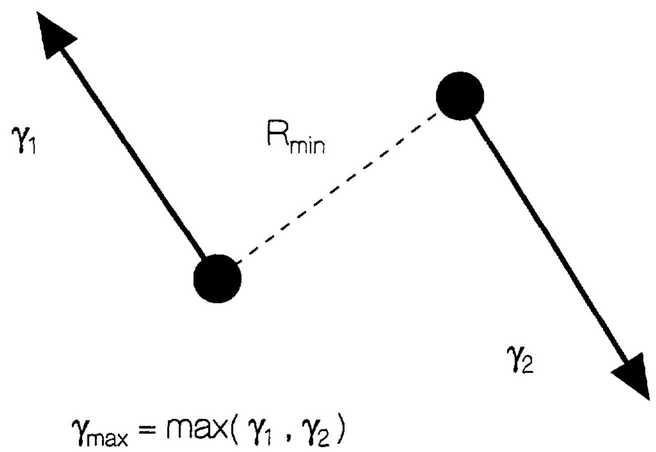



Figure 9.2: The vorton configuration on which time-step adaptation is based (see text).

order 1. A maximum time step $\Delta t_{max}$ was imposed, usually $10^{-4}s$. Both values are based on experience obtained during the performance of the simulations.

Unless otherwise stated, all length scales are in *cm*, and all time scales in *s*.

## 9.3 Diagnostics

Our numerical study of vortex phenomena consists of considering the general, qualitative behaviour of the simulated test cases and comparing it with experimental results. Besides, an investigation will be done regarding some flow diagnostics, i.e. quantitative data which can be visualized either by means of graphs or data visualization packages.

In general, diagnostics can serve three purposes:

1. they may be indicators of the accuracy of the numerical simulations;
2. they may be useful in the comparison of different simulation methods;
3. they may be useful in the comparison of results from simulations with those from experiments.

For our simulations we introduce two kinds of diagnostics: motion-invariants and fields. The first serves the purposes 1 and 2 mentioned above. The second one is meant to serve purpose 3. By means of these diagnostics, we get insight into the behaviour of the cases studied, into the applicability of the proposed vorton method, and into the latter's performance as compared to Novikov's and Kuwabara's vorton method (see §8.3).

### 9.3.1 Motion-Invariants

As for any vortex method, for the vorton method an investigation into the conservation of so-called motion-invariants may be an important test of its accuracy (see §7.2).

In any infinite, inviscid, 3-D flow the following motion-invariants have to be conserved by the flow [6]:

[6] The existence of motion-invariants and their expressions in fluid mechanics can be derived by representing

- **total vorticity:**

$$\boldsymbol{\Omega} \equiv \int_V \boldsymbol{w} \, .$$

$\boldsymbol{\Omega}$ is always conserved for flows with a velocity field approaching zero at infinity at an appropriate rate, since

$$\int_V \boldsymbol{w} = \int_{\partial V} \boldsymbol{v} \times \boldsymbol{n}$$

where $\partial V$ is the (infinitely distant) boundary of volume $V$.

For the original vorton vorticity field (8.5), we get the simple result:

$$\boldsymbol{\Omega} = \sum_{\alpha} \boldsymbol{\gamma}_{\alpha} \, . \tag{9.1}$$

For the divergence-free vorton vorticity field (8.11), the integral related to $\boldsymbol{\Omega}$ is not defined due to the singular nonlocal part. However, Appendix C shows that for a volume $V$ equal to a small sphere around a vorton location $\boldsymbol{r}_{\alpha}$, $\Omega$ is proportional to $\gamma_{\alpha}$. This may suggest that (9.1) remains valid for our vorton method.

However, expression (9.1) is easily shown to be an inappropriate expression for $\boldsymbol{\Omega}$. It can only be conserved if the expression for $\dot{\boldsymbol{\gamma}}_{\alpha}$ is pairwise anti-symmetric for any pair of vortons $\alpha$ and $\beta$. This is only the case for the K-equation (8.21), but not so for the N-equation (8.20) and the N+K-equation (8.22). This shows that our assumption on the validity of (9.1) is not valid. Consequently, we have decided to disregard (9.1) as a motion-invariant.

- **total linear momentum:**

$$\boldsymbol{P} \equiv \int \boldsymbol{v} \, . \tag{9.2}$$

Due to the singular behaviour of the vorton velocity field (8.10) an expression for $\boldsymbol{P}$ cannot be obtained for the integral in expression (9.2).

Expression (9.2) can be rewritten into the form of the so-called Kelvin impulse [7] (see (A.3) in Appendix A):

$$\boldsymbol{P} \equiv \frac{1}{2} \int_V \boldsymbol{x} \times \boldsymbol{w} \tag{9.3}$$

if the vorticity field satisfies the condition

$$\mid \boldsymbol{w}(\boldsymbol{x}) \mid \sim x^{-n} \, , \; n > 3 \;\text{ as }\; x \to \infty \tag{9.4}$$

is satisfied. This condition is not fulfilled by our vorton vorticity field (8.11). If instead we apply the original vorticity field (8.5) to (9.3), we get:

$$\boldsymbol{P} = \frac{1}{2} \sum_{\alpha} \boldsymbol{r}_{\alpha} \times \boldsymbol{\gamma}_{\alpha} \, . \tag{9.5}$$

the Euler or Helmholtz equation in terms of so-called Poisson brackets, related to Lie-algebra (see e.g. [72]).

However, for of a divergence-free vector potential the existence of the first three motion-invariants mentioned below can directly be derived as shown in Appendix A. Though our field (8.8) is not divergence-free, the results provided in Appendix A explain some of the problems in the derivation of motion-invariants given below.

[7] This denomination seems historically incorrect, since this expression has not been traced in Kelvin's work. It has been derived by J.J. Thomson in his *Treatise* (see §5.1).

Since expression (9.1) is unequal to zero, (9.5) shows lack of invariance with regard to the location of the frame of reference. Nevertheless, it will be taken as representation of $\boldsymbol{P}$ in the simulations presented in Chapter 10.

- **total angular momentum**:

$$\boldsymbol{J} \equiv \int_V \boldsymbol{x} \times \boldsymbol{v} \,. \tag{9.6}$$

As for linear momentum, no expression for $\boldsymbol{J}$ can be derived from expression (9.6) due to the singular behaviour of the vorton velocity field (8.10).

Under the same condition (9.4) as mentioned above for $\boldsymbol{P}$, we find (compare (A.5) in Appendix A):

$$\boldsymbol{J} = \frac{1}{3}\int_V \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{w}) \,. \tag{9.7}$$

As for linear momentum, our expression to be used as diagnostic for total angular momentum is derived by applying the original vorton vorticity field (8.5) to (9.7):

$$\boldsymbol{J} = \frac{1}{3}\sum_\alpha \boldsymbol{r}_\alpha \times (\boldsymbol{r}_\alpha \times \boldsymbol{\gamma}_\alpha) \,. \tag{9.8}$$

- **total kinetic energy**:

$$E \equiv \int_V \boldsymbol{v} \cdot \boldsymbol{v} \,. \tag{9.9}$$

Direct calculation of $E$ by substituting the vorton velocity field (8.10) into (9.9) can be achieved (see [283]) if the so-called self-energy $E_0$ of the vortons is subtracted (compare with the case of 2-D point-vortices; see §7.3.1), which in the final result is indicated by the omission of terms $\alpha = \beta$. The remaining part of $E$ is called the interaction-energy $E_i$:

$$E_i = \frac{1}{8\pi}\sum_{\alpha\neq\beta}\{\frac{\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{\gamma}_\beta}{R_{\alpha\beta}} + \frac{(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_{\alpha\beta})(\boldsymbol{\gamma}_\beta \cdot \boldsymbol{R}_{\alpha\beta})}{R^3_{\alpha\beta}}\} \,. \tag{9.10}$$

The same result can be derived by a different approach, which makes use of an expression for the energy spectrum $E(k)$ [8]. The energy spectrum also consists of a self-energy part and an interaction-energy part [9]:

$$E(k) = E_0(k) + E_i(k) \tag{9.11}$$

where

$$E_0(k) = \frac{1}{6\pi^2}\sum_\alpha \boldsymbol{\gamma}_\alpha \cdot \boldsymbol{\gamma}_\alpha \tag{9.12}$$

$$E_i(k) = \frac{1}{4\pi^2}\sum_{\alpha\neq\beta}\{\phi_1(kR_{\alpha\beta})(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{\gamma}_\beta) + \frac{\phi_2(kR_{\alpha\beta})(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_{\alpha\beta})\,(\boldsymbol{\gamma}_\beta \cdot \boldsymbol{R}_{\alpha\beta})}{R^2_{\alpha\beta}}\} \tag{9.13}$$

[8] For details we refer to [6].

[9] This spectrum can be obtained from that for soft vortons derived by Kiya and Ishii in [105] by taking $\sigma \to 0$ (see Appendix B).

where

$$\begin{aligned} \phi_1(z) &\equiv \frac{(z^2-1)\sin z + z\cos z}{z^3} \\ \phi_2(z) &\equiv \frac{(3-z^2)\sin z - 3z\cos z}{z^3} . \end{aligned}$$

Integration of $E(k)$ over all wave numbers $k$ ranging from 0 to $\infty$ again shows that total kinetic energy is only finite if the self-energy $E_0$ is subtracted.

To have a possible indication of the behaviour of $E_0$, we take as a diagnostic:

$$E_0 = \sum_{\alpha} \gamma_\alpha^2 \tag{9.14}$$

which is based on the self-energy spectrum $E_0(k)$ given by (9.12).

- **total helicity**:

$$H \equiv \int_V \boldsymbol{v} \cdot \boldsymbol{w} . \tag{9.15}$$

As discussed in §C of the Interlude, helicity is related to the topology of vortex structures in a flow (see e.g. [161]). Besides, $H$ is a fundamental motion-invariant of inviscid flows.

For the integral (9.15) to be convergent, it is sufficient that the vorticity field $| \boldsymbol{w}(\boldsymbol{x}) | \sim x^{-4}$ as $x \to \infty$ for a fluid of infinite extent, in order to ensure invariance of $H$ [157]. The vorton vorticity field (8.11) does noy satisfy this requirement.

However, the above condition is not a *necessary* one and an expression for $H$ can be derived from our vorton fields without objections. As for total kinetic energy, helicity has to be split into a self-helicity part and an interaction-helicity part. Inserting the vorton velocity (8.10) and vorticity (8.11) fields in (9.15), we get for the interaction-helicity:

$$H_i = \frac{1}{4\pi} \sum_{\alpha,\beta} \frac{\boldsymbol{R}_{\alpha\beta} \cdot (\boldsymbol{\gamma}_\alpha \times \boldsymbol{\gamma}_\beta)}{R_{\alpha\beta}^3} . \tag{9.16}$$

Though not a motion-invariant in the same sense as those listed above, we also have to regard the diagnostic provided by the circulation $\Gamma$. In §4.1 Kelvin's derivation of the circulation concept (4.2) and its most important property, the circulation theorem, were presented. From this theorem it follows that, as for a vortex ring, the circulation of a vorton ring should be conserved.

To calculate a value for $\Gamma$ for the vorton ring, a closed curve $C$ will be chosen as indicated in fig.9.1 along which the velocity field (8.10) will be integrated. This will be done by taking the value of the tangential velocity component at equally spaced grid points along the curve, multiplying this value by the local distance between the points, and adding all contributions.

Novikov [168] has shown analytically that in the limit of infinite number of vortons, i.e. $N \to \infty$, the circulation of a vorton ring of radius $R$ can be written as:

$$\Gamma = \frac{\gamma}{\frac{2\pi R}{N}} \tag{9.17}$$

where $\gamma$ is the strength of each vorton in the ring and the denominator can be regarded as the distance between the vortons in the ring. Despite the relatively small values for $N$ in our simulations, expression (9.17) has been used in the numerical simulations to obtain the value of the vorton strength $\gamma$ for the vortons in a vorton ring of given radius $R$, circulation $\Gamma$, and number of vortons $N$.

### 9.3.2 Fields

The following fields will be regarded:

- velocity field

  The velocity field $\boldsymbol{v}$ is given by expression (8.10).

- vorticity field

  The vorticity field used as a diagnostic is the nonlocal gradient part of the divergence-free vorton vorticity field (8.11). To stress the point that this field is only part of the divergence-free vorticity field (8.11), it will be indicated by $\bar{\boldsymbol{w}}$, i.e.:

$$\bar{\boldsymbol{w}}(\boldsymbol{x}, t) \equiv \sum_{\alpha} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \frac{\boldsymbol{\gamma}_\alpha}{R_\alpha}) . \tag{9.18}$$

The fields are calculated on the points of a rectangular grid which is constantly adapted to the vorton configuration. Due to computational restrictions the grid is usually limited to $40^3$ grid points.

Notice that both fields show singular behaviour at the vorton locations. To avoid problems coincidence of the grid points with vorton locations has been prohibited by either shifting the grid or nelgecting the contribution of some vortons during the calculation of the fields.

Visualization of the vorticity field, by means of isosurfaces of its magnitude $|\ \bar{\boldsymbol{w}}\ |$, has been done by means of the graphics package AVS. Some figures show the vortons themselves (as arrows, consisting of a line and a open circle as arrow head). These have been made by means of the specially-written package Vectrix. In these pictures the vorton strength vectors have been scaled to the same length.

## 9.4 Vorton Division

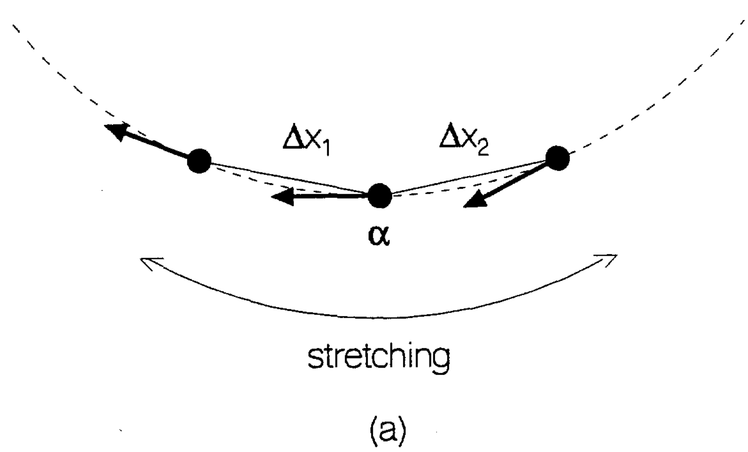
One of the seemingly attractive properties of the vorton method is the possibility of adding and removing vortons. The addition of vortons, to be called vorton division, has been introduced by Kuwabara (see e.g. [113]) and also has been applied by Pedrizetti [178] and Winckelmans [283].

Its principle is illustrated in fig.9.3: if the distance $\Delta x = \Delta x_1 + \Delta x_2$ of a vorton $\alpha$ to its two nearest neighbours has increased beyond a certain value $\lambda$ times the original distance $\Delta x_0$, vorton division will be imposed [10].

For the authors mentioned above, division meant the removal of vorton $\alpha$ and addition of two new vortons at locations $\pm(\Delta x/8)\ \boldsymbol{\gamma}_\alpha/\gamma_\alpha$. In several cases this procedure will lead to an irregular distribution of vortons in azimuthal direction, e.g. in the case of a radially growing vorton ring (as we will encounter in §10.3). Therefore, we have adopted another vorton division procedure: two new vortons will be added in between vorton $\alpha$ and its neighbours, while vorton $\alpha$ is left untouched; see fig.9.3(b).

Various options could be proposed for assigning vorton strengths to the added vortons and for optional updating of the strengths of the existing vortons. The simplest choice is to

[10]Discussion of the value to be given to $\lambda$ will be postponed to §10.3.2.



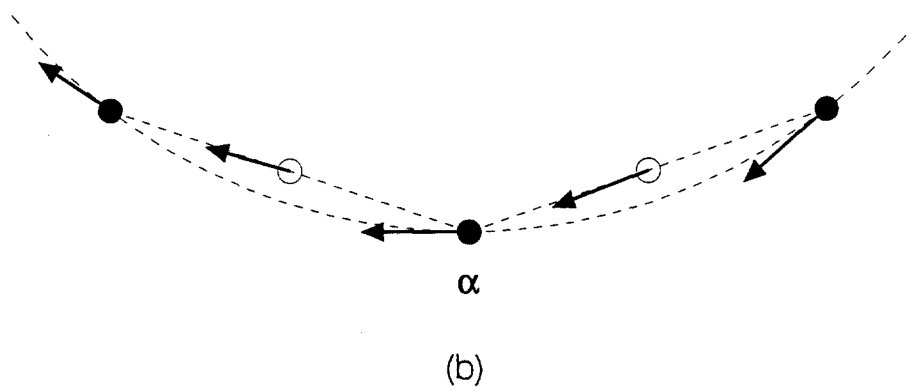




Figure 9.3: Illustration of the vorton division procedure (see text): (a) before ($\Delta x_0 = \Delta x(t=0) = \Delta x_1 + \Delta x_2$) and (b) after division ($\Delta x = \lambda \Delta x_0$). Added vortons are indicated by open dots.

take for the strength of each added vorton the mean value of its two neighbours in the ring and to omit updating of the existing vortons. However, in that case no attention has been given to the influence of vorton division on the conservation of the expressions for the motion-invariants mentioned in §9.3.1. Our choice of updating the vortons after division is based on the conservation of circulation [11] as defined by (9.17). This means that for every vorton the quantity

$$\frac{\gamma}{\Delta x} \tag{9.19}$$

is to be conserved, where $\gamma$ is its strength and $\Delta x$ is defined in fig.9.3(a). This procedure will be called "division with updating".

Summarizing, our division procedure is as follows:

1. new vortons are added as indicated in fig.9.3 whenever $\Delta x$ becomes larger than $\lambda \Delta x_0$;

[11] Winckelmans [283, §3.5.7] has suggested that division procedures can be set up which conserve both linear momentum as given by expression (9.5) and angular momentum given by (9.8). However, two remarks have to be made. First, we have indicated in §9.3.2 that these expressions cannot be proper motion-invariants in case of the vorton method as applied here. Secondly, one could wonder whether conservation of both these invariants assures the conservation of other motion-invariants.

2. the strengths of the existing vortons are updated such that the value of expression (9.19) becomes equal to the value it had just before the division started;

3. the newly added vortons get strengths which are the average of the strengths of their two neighbours;

4. the former values of $\Delta x_0$ are updated according to the new distances between the vortons.

# Chapter 10

# Numerical Simulations: Results

In this chapter we present the results of our numerical simulations for the six test cases mentioned in §9.1. Besides, relevant results from literature are presented for comparison and evaluation. The description of the results is sometimes accompanied by concluding remarks, though the main conclusions and general discussion of our results, and the vorton method in general, are postponed to Chapter 11.

## 10.1 Single Vorton Ring

In this section we compare the properties and characteristics of a single vorton ring, as illustrated in fig.9.1. In §10.1.1 we treat the velocity and vorticity distribution inside the core of the ring and its velocity of translation. In §10.1.2 the stability of vorton rings is discussed.

### 10.1.1 General Characteristics of the Vorton Ring

As mentioned in §9.1, the vorton ring is determined by four parameters: radius $R$, circulation $\Gamma$, strength $\gamma$ of each vorton, and the number of vortons $N$. For given $R$ and $\Gamma$, a relation between $\gamma$ and $N$ can be derived from relation (9.17).

In fig.10.1 a standard vorton ring ($N = 12$) has been visualized by means of isosurfaces of the magnitude of diagnostic $\bar{\boldsymbol{w}}$ given by (9.18). Fig.10.1(a) shows that the vorton ring has a core. However, the contribution of each vorton remains detectable, which becomes clearer if the value of $|\, \bar{\boldsymbol{w}} \,|$ is increased as shown by fig.10.1(b). Obviously, for larger number of vortons, the isosurfaces will become smoother.

In fig.10.2 the distribution of velocity and vorticity is shown for the core of a standard vorton ring along the lines indicated in fig.10.2(a). To avoid the singularity in both fields, the contribution of the vorton on line $A$ has been disregarded. Comparison with the only experimental measurements known, from Maxworthy [149], shows reasonable qualitative agreement. Curve fitting has shown that the distribution of fig.10.2(b) can be described by a curve of the form $\text{sech}^2(r)$ with $r$ the radial distance from the core center which is the curve that has been proposed by Maxworthy.

The distances between the two peaks (maximum and minimum) in the velocity distribution as given in fig.10.2(c), can be taken as a measure for the core size $a$. In fig.10.3 the ratio between the non-dimensional core radius $\tilde{a}$, given by:

$$\tilde{a} \equiv \frac{a}{2\pi R/N} , \tag{10.1}$$

has been plotted as function of the number of vortons $N$ and for the two lines across the ring shown in fig.10.2(a). In both cases, $\tilde{a}$ appears to converge towards a constant value. We conclude that for the standard vorton ring

$$a \propto \frac{R}{N} \text{ as } N \to \infty . \tag{10.2}$$

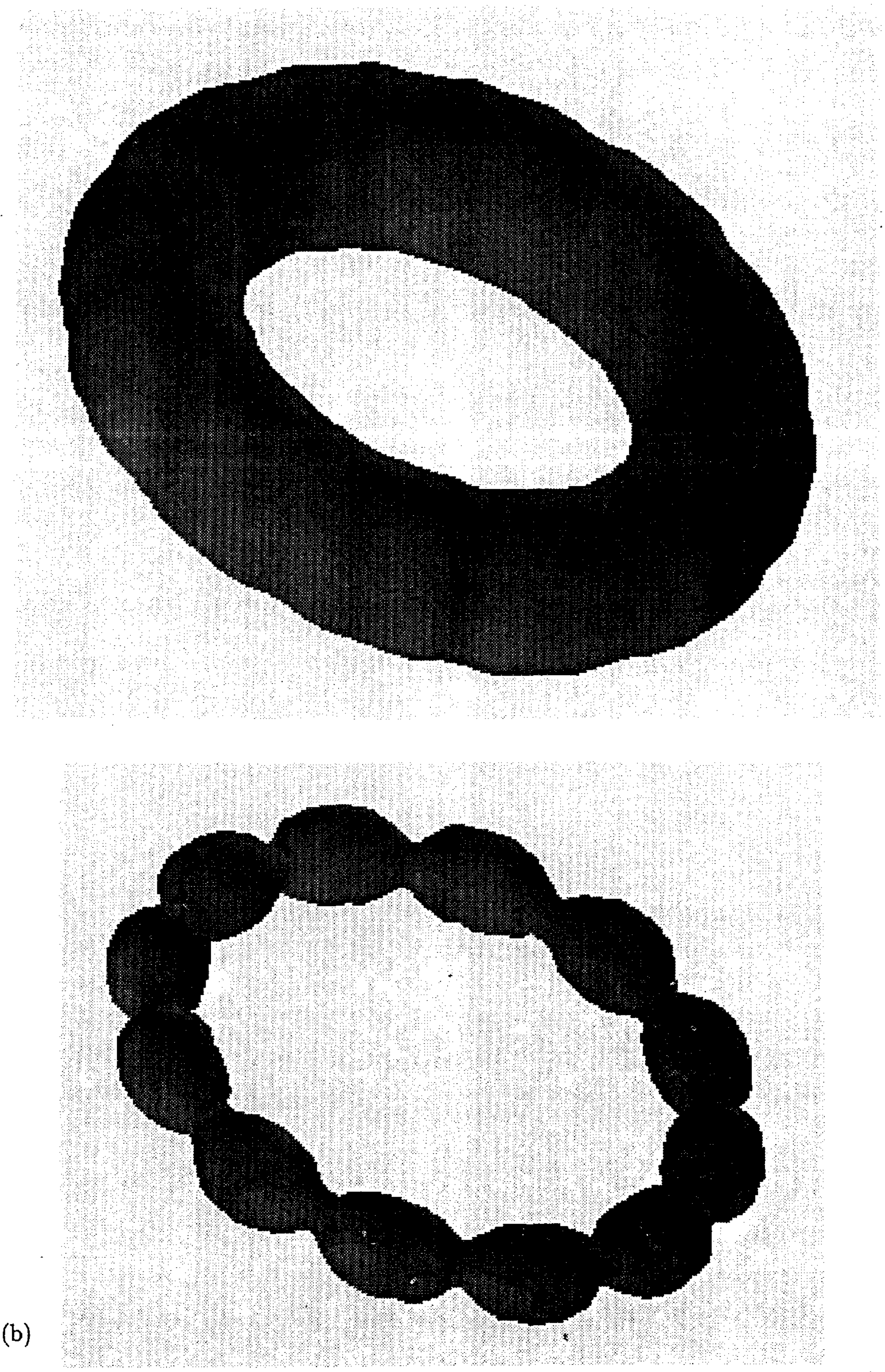

(a)
(b)

Figure 10.1: Single standard vorton ring ($N = 12$): isosurfaces of $|\bar{\mathbf{w}}|$ (see (9.18)) = (a) 1,000 $1/s$, (b) 10,000 $1/s$.

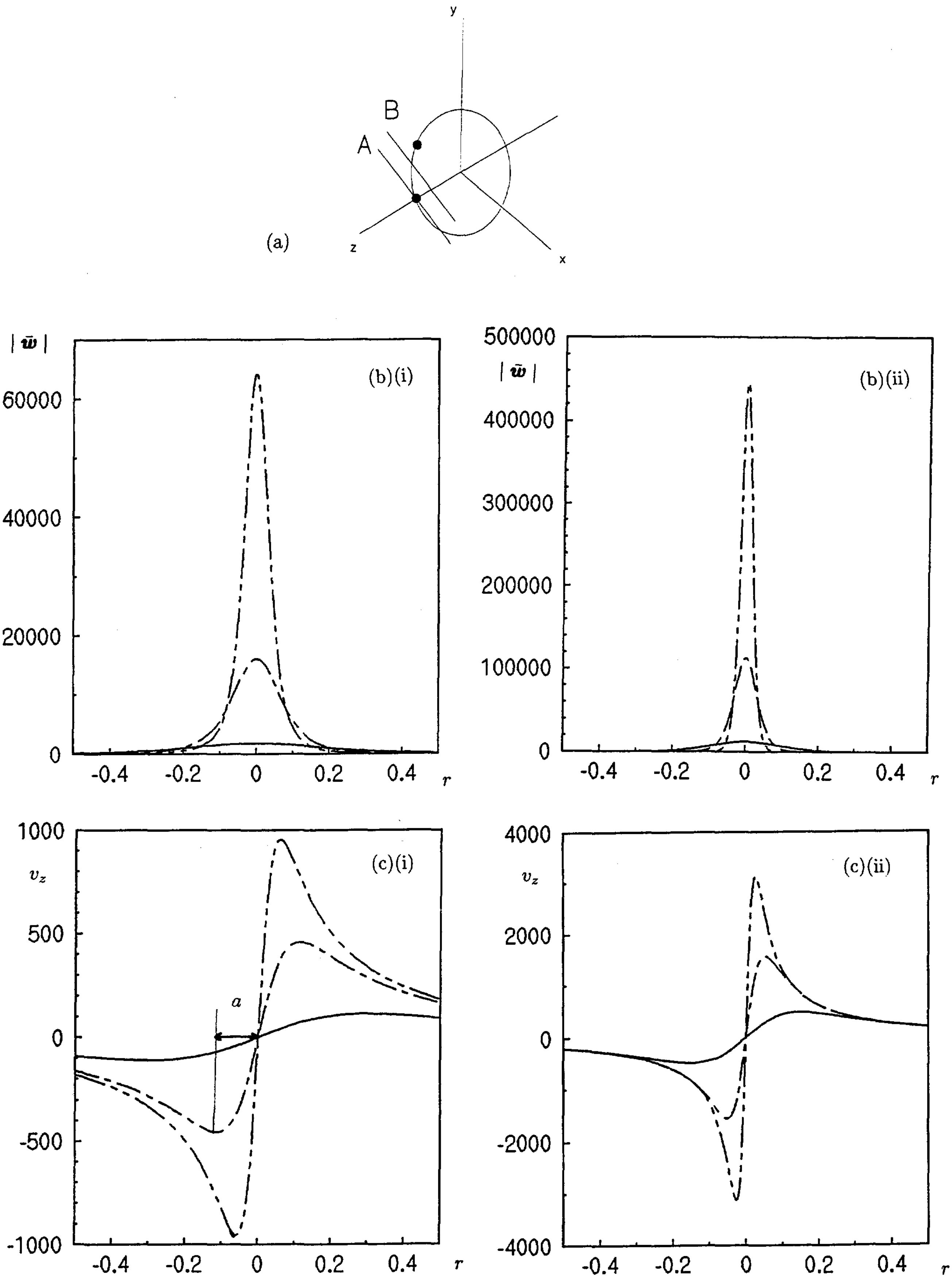



Figure 10.2: Single standard vorton ring (variable $N$): distribution of velocity and vorticity around the center line. (a) situation sketch (the dots indicate vorton locations on the vorton ring); (b) distribution of $|\bar{\mathbf{w}}|$ (given by (9.18)) along (i) line $A$ and (ii) line $B$; (c) distribution of velocity $v_z$ along (i) line $A$ and (ii) line $B$. Number of vortons $N = 12$ (—), 36 (- - -), 72 (- - - -). $r$ = distance along lines $A$ and $B$. The contribution of the vorton on line $A$ has been neglected. $a$ is core radius.

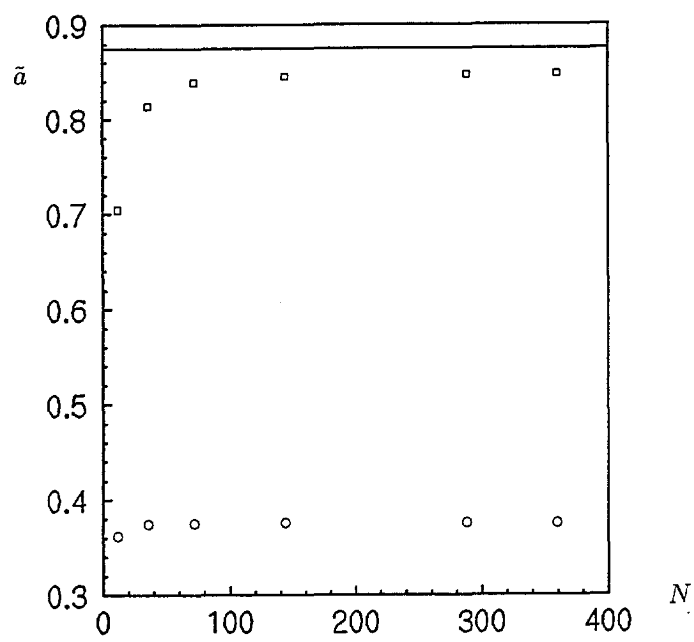



Figure 10.3: Single standard vorton ring: non-dimensional core radius $\tilde{a}$ (given by (10.1)) vs. number of vortons $N$. Core size $a$ determined according to velocity distribution (see fig.10.2(c)) along line $A$ (□) and $B$ (○) (see fig.10.2(a)). The horizontal line $\tilde{a} \approx 0.875$ indicates the value derived from expression (4.3) for a Kelvin-ring.

However, the factor of proportionality appears to depend on the azimuthal location in the ring. In fig.10.3 we have also indicated the value of $\tilde{a}$ which follows for large $N$ from Kelvin's expression (4.3) for the Kelvin-ring (see §A.2 of the Interlude); it appears to be $\approx 0.875$ [1].

Fig.10.1 and figs. 10.2(b)(i) and (ii) show that the vorton ring has an azimuthally inhomogeneous distribution of vorticity. A large value for $N$ can render a more homogeneous distribution in the ring. However, the value $N$ can not be chosen at random when the circulation $\Gamma$, the radius $R$, and the velocity $V$ of a ring are prescribed. This can be seen from the expression for $V$ which can be derived from (10.2) and the general expression for the velocity of a vortex ring given in §A.2 of the Interlude:

$$V = \frac{\Gamma}{4\pi R}(\log N + A' - \frac{1}{2}) \ . \tag{10.3}$$

The factor $A'$ depends on the factor of proportionality in (10.2) and on the vorticity distribution in the core, i.e. on $N$. Hence, the dependence of $A'$ on $N$ is ambiguous and complicated. In the simulations presented in the next sections, the determination of $N$ has been done by taking the (integer) value of $N$ which gives a velocity closest to the prescribed one.

We could wonder whether a change in the number of vortons also means a change in other properties of the vorton ring besides its core size. One indication may be gained from a calculation of the energy spectrum. In fig.10.4 the energy spectrum of a vorton ring ($R = 1$, $\Gamma = 1$) given by (9.11) is shown for two values of $N$.

[1] This value has also been found by Pedrizzetti [177].

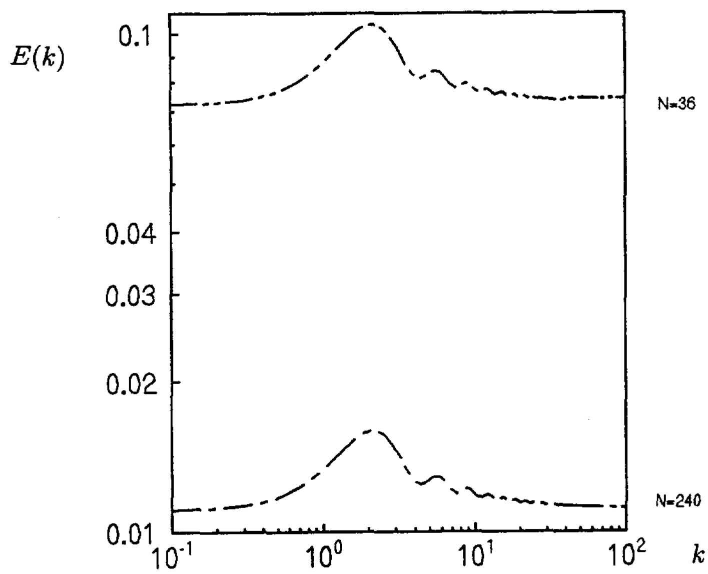



Figure 10.4: Single vorton ring ($R = 1, \Gamma = 1$): energy spectra $E(k)$ according to (9.11) for number of vortons $N = 36$ and $N = 240$. $k$ is wavenumber.

We observe that the shape of the spectra is identical for both values of $N$; they are only shifted along the vertical axis. Apparently, the number of vortons does not influence the physical character of the vorton ring. However, with regard to the shape of the spectrum itself it is hard to make any remarks. According to the analytical result presented in [121] for a "coreless" vortex ring, the spectrum of a ring consists of two parts: a $k^2$ behaviour for small $k$ is due to the nonzero impulse of the ring; a $1/k$ behaviour for large $k$ is that of an isolated smooth vortex tube with $ka \ll 1$ where $a$ is the core size. We could conclude from fig.10.4 that our vorton ring is not "coreless". Unfortunately, we have no information on the exact definition of this vortex ring and other (experimental) results have not been found.

### 10.1.2 Stability of the Vorton Ring

In §A.3 of the Interlude, the development of research on the stability of vortex structures (especially vortex rings), stimulated by Kelvin's vortex atom theory, has been reviewed. The work by Widnall and others on the stability of vortex rings has shown that even in the inviscid case instability can set in [2].

Numerical investigation of the stability of vortex rings has been performed by Knio & Ghoniem [108]. They used a vortex method which can be compared to the soft-vorton method (see Appendix B).

We have simulated the same rings as those used by Knio & Ghoniem, i.e. radius $R = 1$ and circulation $\Gamma = 2$. The initial disturbance has been a sinusoidal radial disturbance, given by:

$$\Delta R \sin(n\theta) \tag{10.4}$$

[2] The influence of viscous effects on the stability behaviour of vortex rings might be negligible, as is reported in e.g. [128].

where $\Delta R$ is the amplitude of the disturbance, $n$ is the wavenumber and $\theta$ is the azimuthal coordinate (see fig.c of the Interlude for an illustration). For $\Delta R$ the value $0.02 * R$ has been chosen. The value of $n$ has been varied and the value at which the amplitude of the disturbance increased fastest, is called $n^*$. In fig.10.5 the most unstable wavenumbers $n^*$ are plotted against the non-dimensional velocity of the ring $\tilde{V}$, given by:

$$\tilde{V} \equiv V \frac{4\pi R}{\Gamma} \tag{10.5}$$

where $V$ is the ring velocity, calculated directly from the displacement of the vorton ring. The

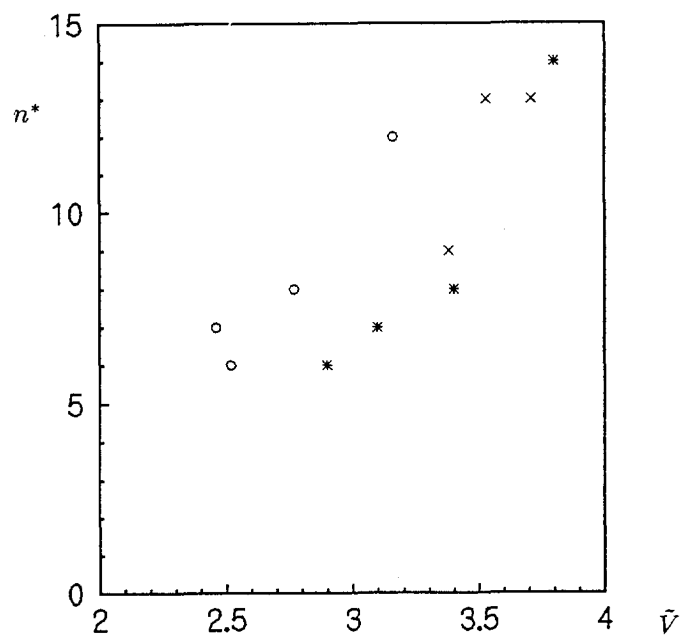



Figure 10.5: Single vorton ring ($R = 1, \Gamma = 2$): most unstable wave number $n^*$ vs. the non-dimensional ring velocity $\tilde{V} = V(4\pi R/\Gamma)$. Results for ($\times$) the vorton ring are compared to ($*$) numerical results from [108] , and ($\circ$) experimental results from [278] .

amount of data may seem small. Unfortunately, the range of values of $\tilde{V}$ for which Knio & Ghoniem provide data is in a region where, for $\Gamma = 2$, the number of vortons in the vorton ring is about twice the number of waves $n$. This means that the representation of the sinusoidal disturbance is inaccurate with unknown effect on the instability behaviour of the vorton ring.

Nevertheless, we conclude that the stability behaviour of the vorton ring compares reasonably well with the numerical results of Knio & Ghoniem. However, both our and Knio & Ghoniem's results do not compare very well with experimental data as given by Widnall & Sullivan [278], also indicated in fig.10.5.

## 10.2 Behaviour of a Single Pseudo-Elliptical Vorton Ring

### 10.2.1 Introduction

The dynamics of a single vortex ring becomes much more interesting (and complicated) when its shape is changed from circular to elliptical. From accounts of the smoke ring experiments performed by Kelvin and Tait in the 1860s, it appears that they already recognized the peculiar behaviour of elliptical rings (see §3.2). However, only in recent years this behaviour has been studied more deeply.

The behaviour of an elliptical ring is determined by its inclination to "oscillate", i.e. to constantly interchange its long and short axis. This can be ascribed to the variations in curvature, which induces unequal velocities along the circumference of the ring. The parts of largest curvature will move forwards at a higher velocity than the other parts of the ring, causing a bending of the ring perpendicularly to its plane. The parts staying behind will start to move outwardly, causing the the change of axes, mentioned above. The oscillating behaviour is clearly exposed in fig.10.6.

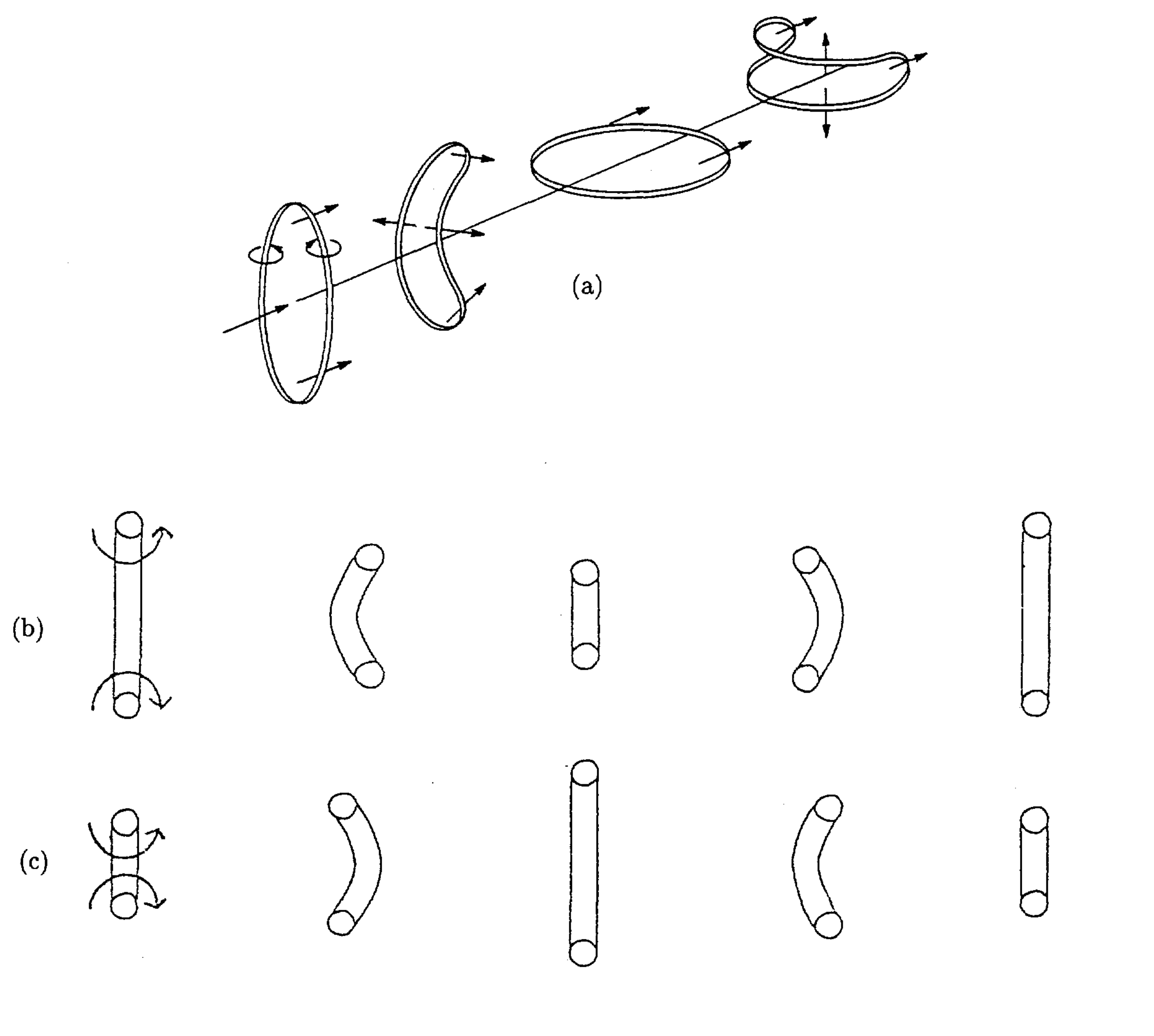



Figure 10.6: Oscillating behaviour of an elliptical vortex ring: (a) perspective view (sketch from [134]); (b) side-view; (c) top-view. Development in time is from left to right.

It has been shown experimentally that the behaviour of elliptical rings depends on the ratio of the lengths of the major and minor axis of the ellipse. When this ratio exceeds a certain value the periodic oscillating behaviour of fig.10.6 is "disturbed" due to vortex reconnection.

Kiya & Ishii [105] performed numerical simulations applying a soft-vorton method and did experiments on the behaviour of a so-called pseudo-elliptical vortex ring whose shape is shown in fig.10.7. This ring is supposed to show similar behaviour as purely elliptical rings. They

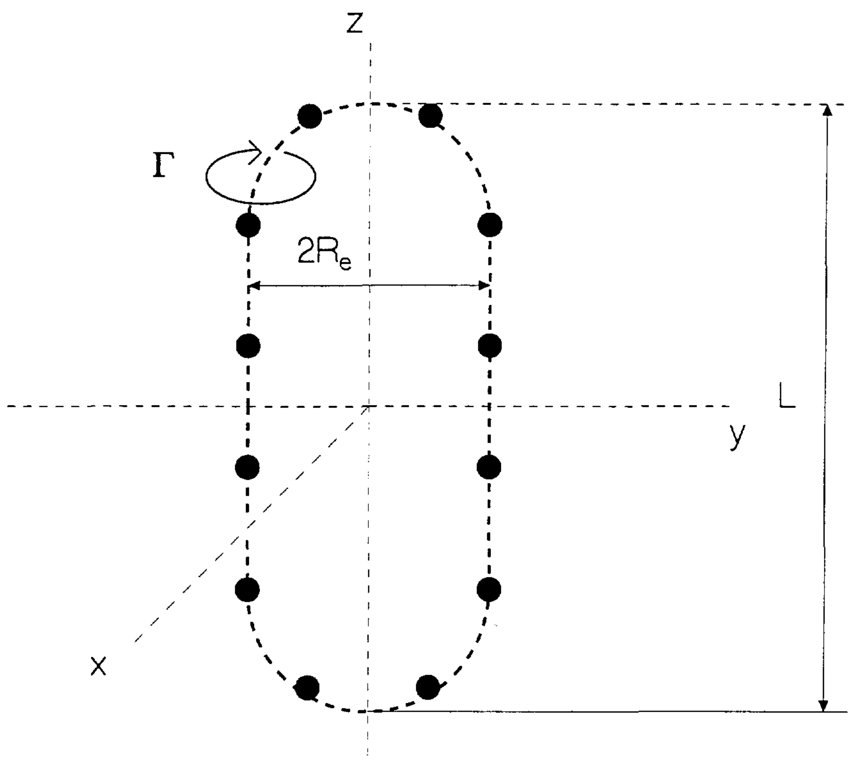



Figure 10.7: A pseudo-elliptical vorton ring. The dots indicate the vorton locations.

observed at least two different regimes in the behaviour of these rings, depending on the axis ratio $L/2R_e$:

1. for $2 < L/2R_e < 4$: oscillation as in fig.10.6 with a continuous interchange of the position of the long and short axis;

2. for $5 < L/2R_e < 8$: the ring's axes interchange one time (as in the first half of fig.10.6)(b); this is followed by splitting up into two rings.

As remarked above, the splitting behaviour may be attributed to the phenomenon of reconnection, introduced in the Interlude [3]. When the long straight parts of the ring approach each other under certain conditions, they are cut and reconnected as illustrated in fig.e of the Interlude.

### 10.2.2 Vorton Simulations

For our numerical simulations, we have taken the vorton representation of Kiya & Ishii's pseudo-elliptical ring (see fig.10.7). For this ring circulation $\Gamma = 225$ and $R_e = 1$. By changing the value of $L$ we can change its axis ratio.

[3] We shall return to this topic in §10.4.

The behaviour of this ring with $N = 32$ and $L/2R_e = 7$ is shown in fig.10.8 in case of application of all three vorton deformation equations, i.e. the N-, K-, and N+K-equations. In case of the K-equation the simulation had to be stopped after a short time as the vorton strengths of some vortons increased dramatically. Time step reduction did not show improvement in this case. The other two cases (figs. 10.8 (a) and (c)) show qualitatively similar behaviour.

In fig.10.9 the development of the the x-component of linear momentum $\boldsymbol{P}$ according to (9.5), interaction-energy $E_i$ according to (9.10), and self-energy $E_0$ according to (9.14) are shown for the three cases of fig.10.8. The results for the K-equation show severe violation of conservation of $\boldsymbol{P}$ and $E_0$. We observe that qualitatively the N-equation and the N+K-equation show no large differences, so that no preference for one of these can be expressed based on these results. However, the results on the conservation of $\boldsymbol{P}$ and $E_i$ suggest that the N+K-equation performs slightly better.

The influence of the number of vortons $N$ on the diagnostics has been investigated in case of the N+K-equation. In fig.10.10, the development of the same diagnostics as in fig.10.9 are shown for two values of $N$ and an axis ratio of $L/2R_e = 7$. We again observe that the invariants are reasonably well conserved. For increasing $N$ their deviations from perfect conservation decreases. Furthermore, we note that the curves of interaction-energy $E_i$ and of self-energy $E_0$ show opposite behaviour.

Regarding the two different regimes of behaviour mentioned above, for our pseudo-elliptical ring (in case of the N+K-equation) we found that regime 1 occurs for $L/2R_e < 8$. Regime 2 does not occur, though in some range of $L/2R_e$ the behaviour seems close to reconnection (as can be observed in fig.10.8(c)). An increase of the numbers of vortons $N$ showed an inclination towards regime 2 for the axis ratios found by Kiya & Ishii. However, above a certainvalue of $N$ the initially straight parts of the pseudo-elliptical rings became unstable and the ring collapsed.

The disagreement between our numerical results and the results given by Kiya & Ishii has been confirmed by a comparison of our numerical results with the results from a simple experiment which we performed with smoke rings in the manner of Tait's 1867 experiment discussed in §3.2 [4]. In fig.10.11 we show both the isosurfaces of the magnitude of diagnostic $\bar{\boldsymbol{w}}$ given by (9.18) for our pseudo-elliptical vorton ring ($N = 8$, N+K-equation) and photographs of the experimentally observed smoke rings. In both cases the axis ratio $L/2R_e = 7$.

One possible explanation for the disagreement between vorton simulation and experiment is the presence of a slight restriction in the middle of the straight part of the smoke ring caused by the way of generation. We have tried to simulate this restriction as indicated in fig.10.12. For a slightly disturbed pseudo-elliptical vorton ring [5], reconnection indeed occurred for $L/2R_e = 7$. However, shortly after the splitting, the two rings linked to become one ring again; see fig.10.12. Apparently, the cause for the inability of our vorton simulation to show

[4]For a full description of our experiment, we refer to [44].

[5]The exact nature of the restriction appeared to be irrelevant. Fig.10.12 shows a simulation in which a sinusoidal disturbance on the straight parts of the ring has been imposed (see the sketch).

Figure 10.8: (*see inserted sheets*) Single pseudo-elliptical vorton ring ($N = 32, L/2R_e = 7$): behaviour in case of (a) N-equation, (b) K-equation, (c) N+K-equation. Three views of the ring are given at each time $t$.

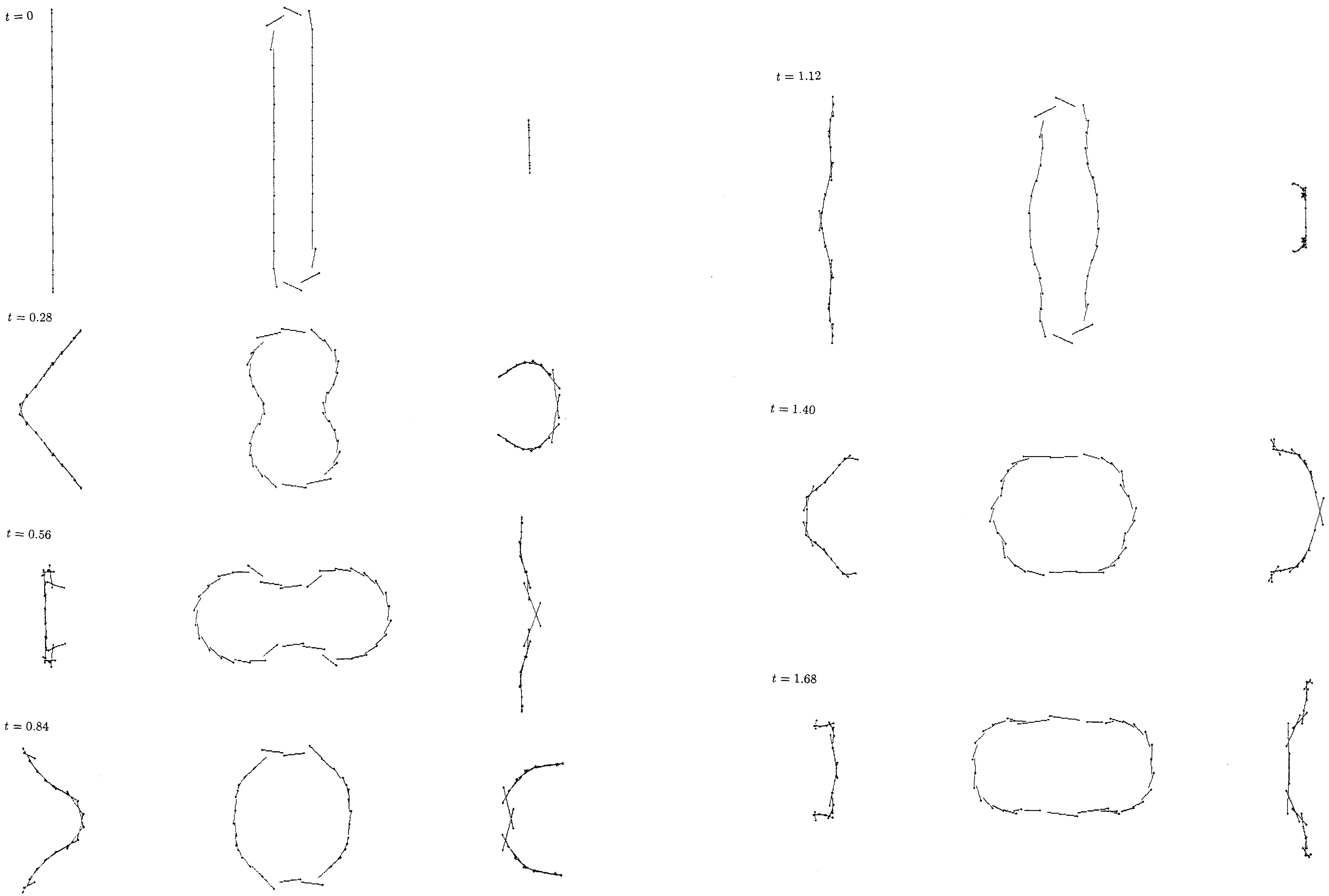



Figure 10.8 (a)

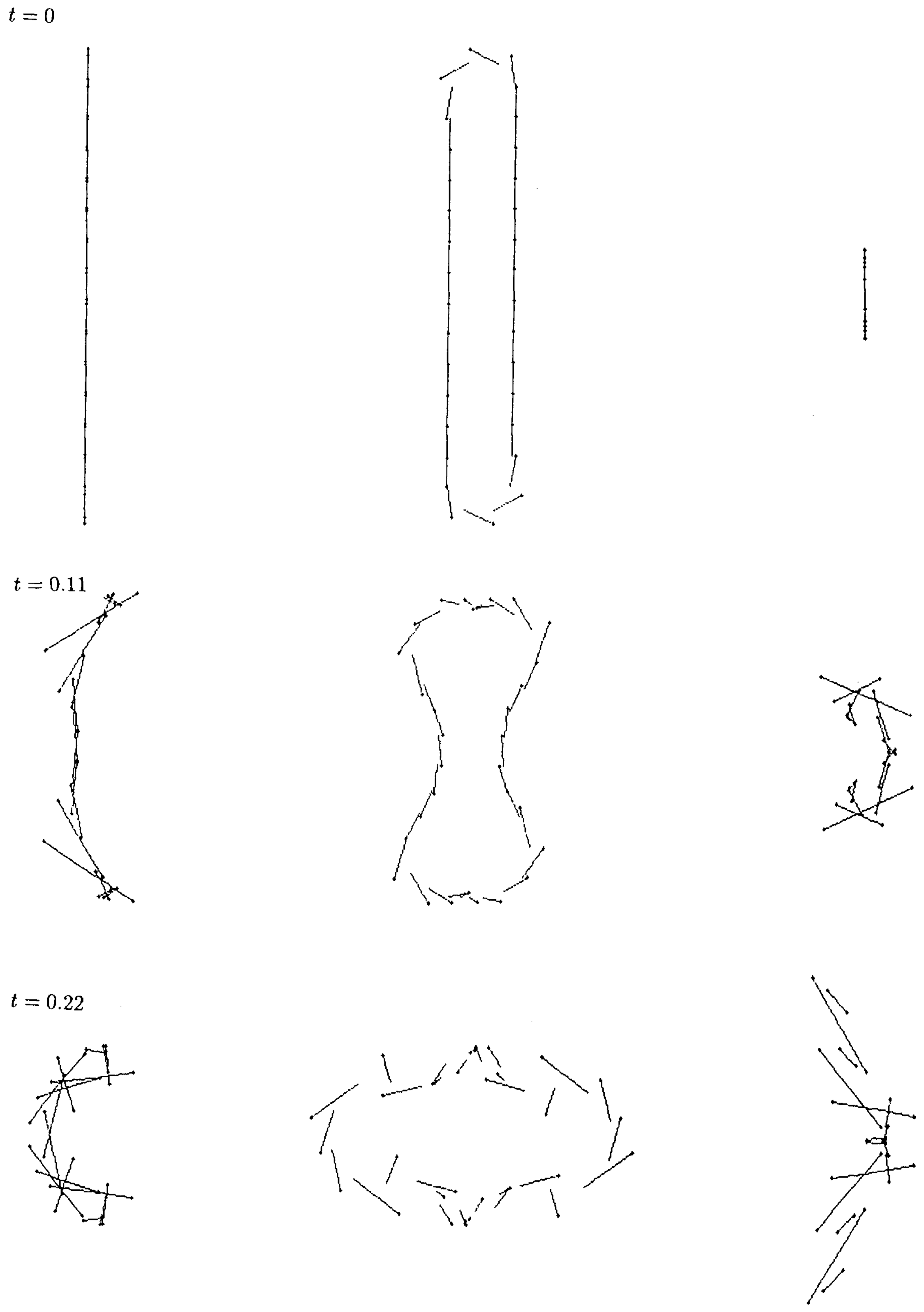



Figure 10.8 (b)

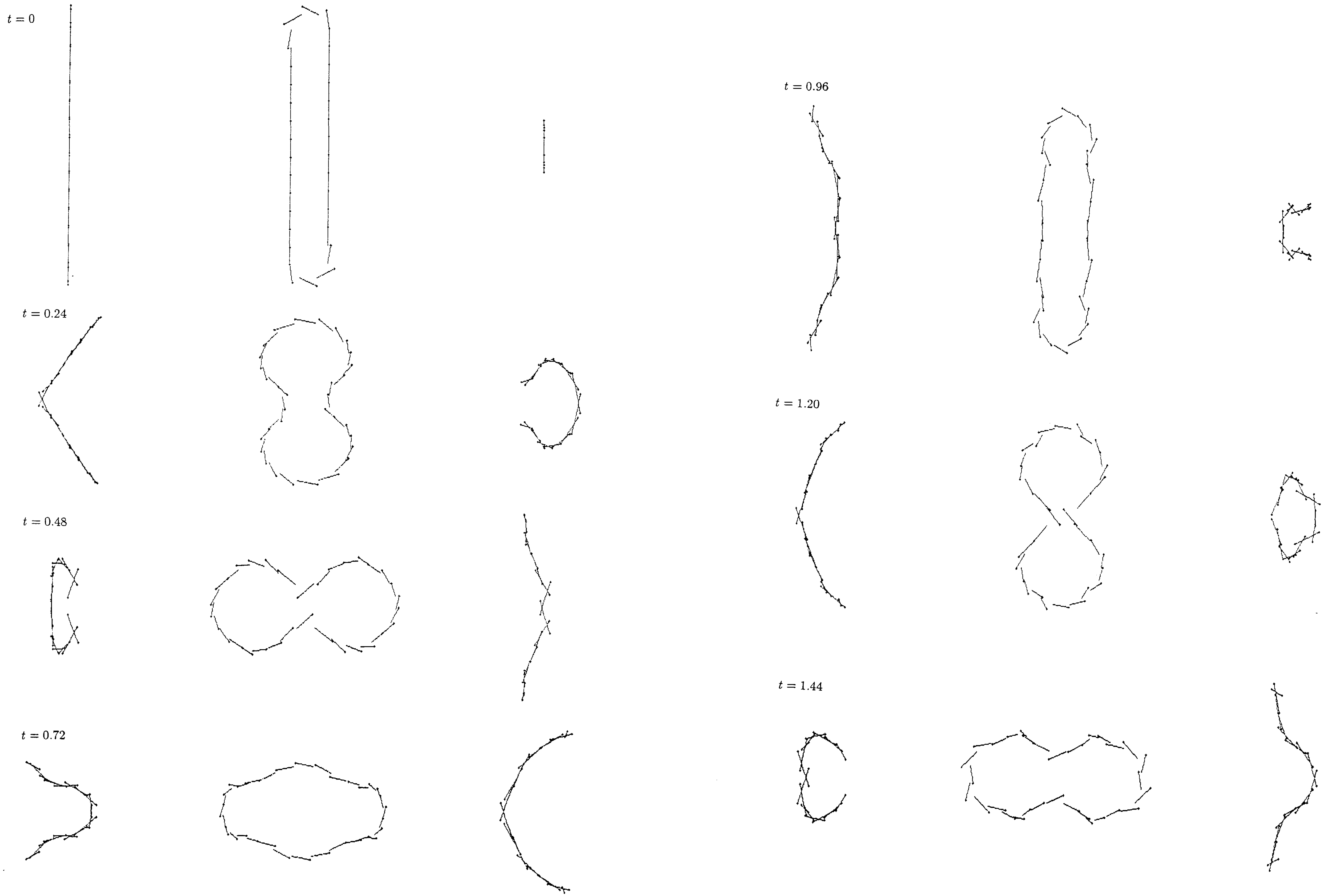



Figure 10.8 (c)

(a)

(b)

(c)

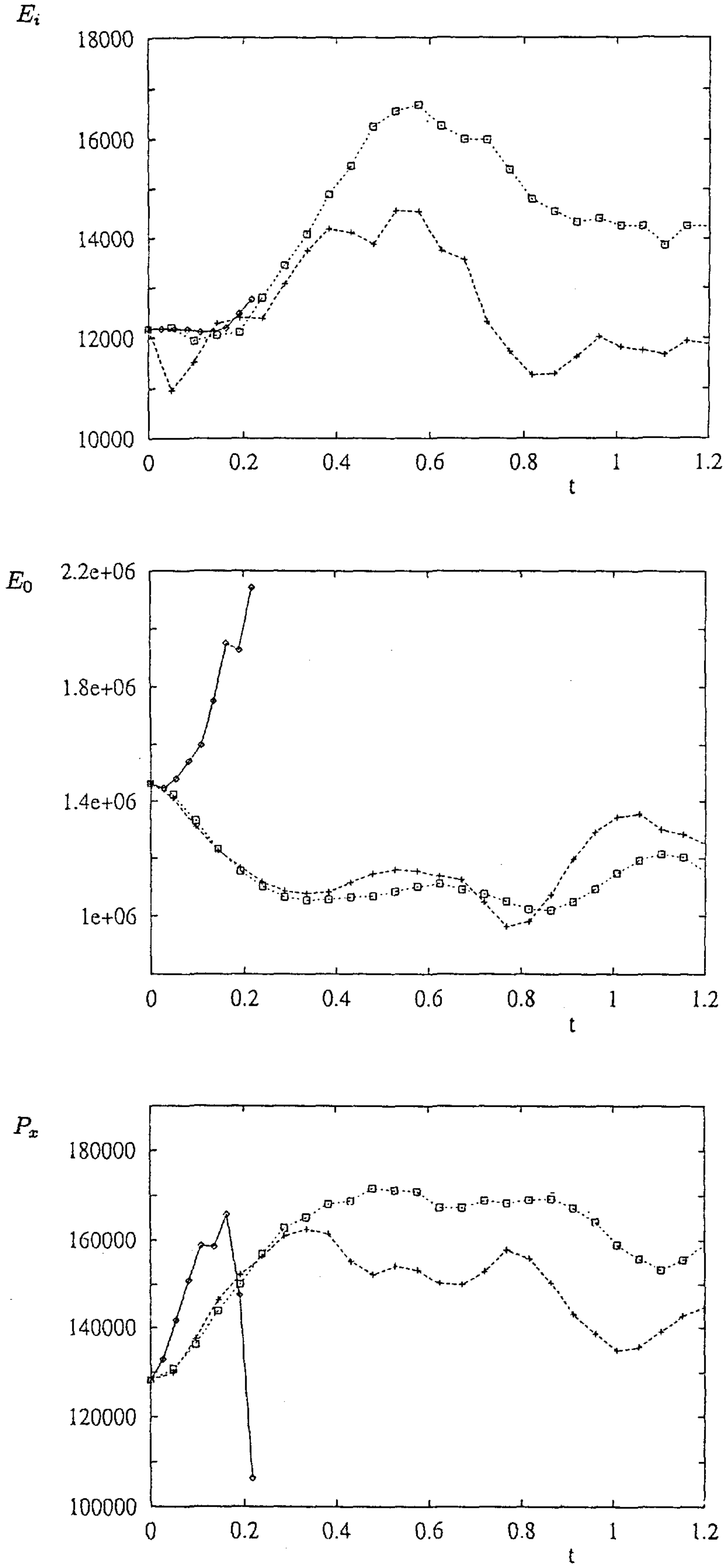
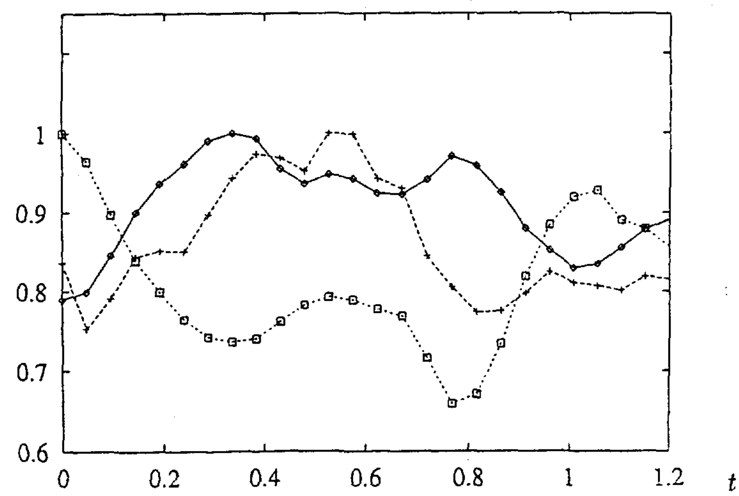



Figure 10.9: Single pseudo-elliptical vorton ring ($L/2R_e = 7, N = 32$): (a) interaction-energy $E_i$ according to (9.10), (b) self-energy $E_0$ according to (9.14), (c) x-component of linear momentum $\mathbf{P}$ according to (9.5). Vorton deformation according to (□) N-equation, (◇) K-equation, (+) N+K-equation. $t$ is time.

(a)




(b)

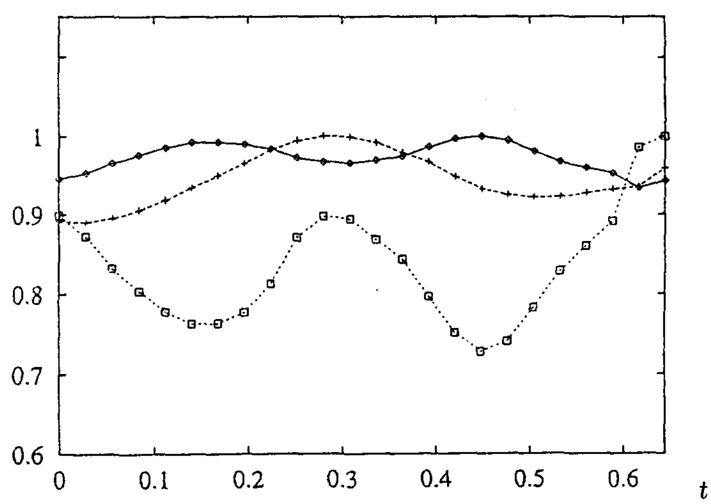
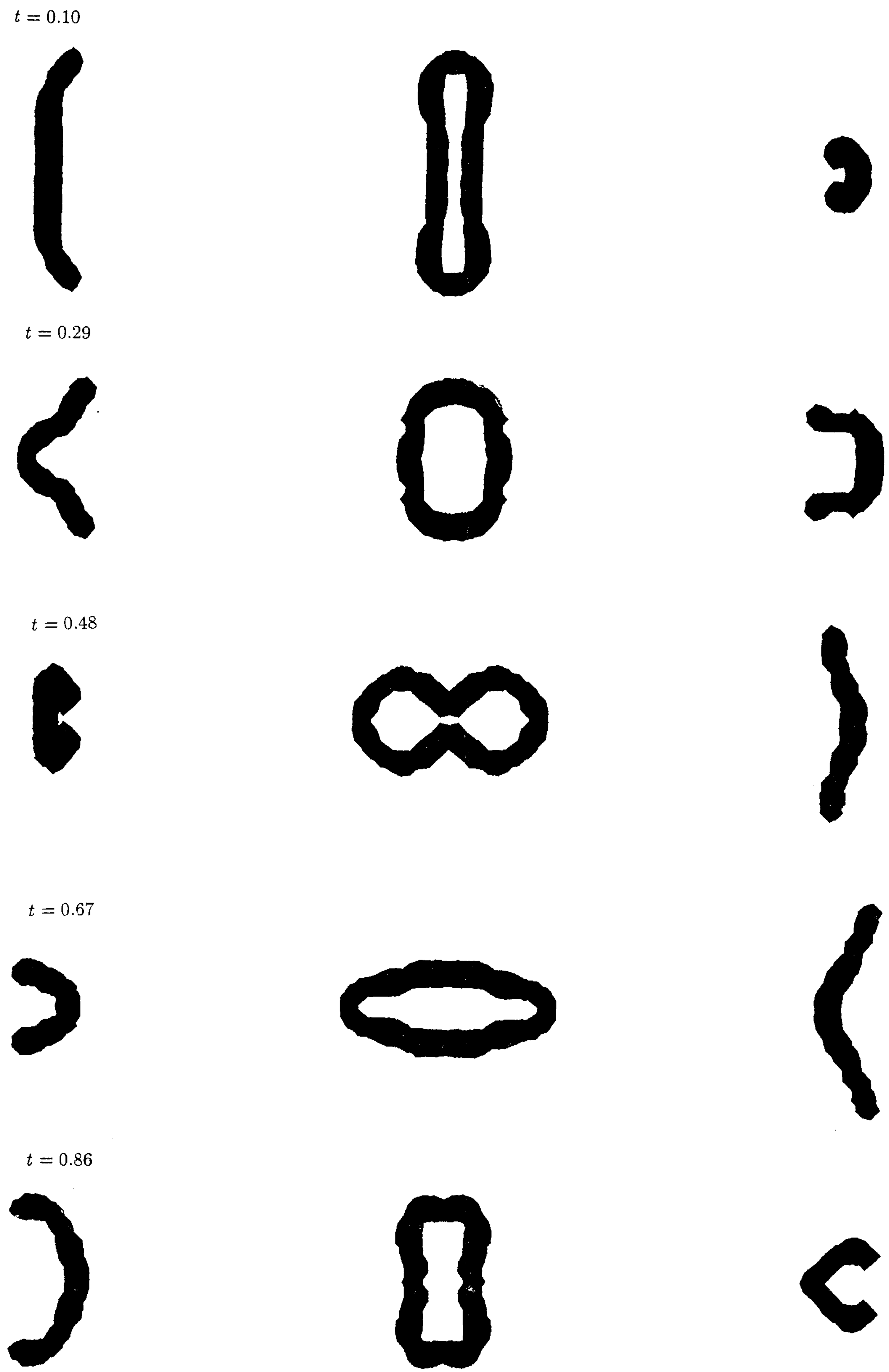



Figure 10.10: Single pseudo-elliptical vorton ring ($L/2R_e = 7$) consisting of $N =$ (a) 32, (b) 192 vortons: (+) x-component of linear momentum **P** according to (9.5), (◇) interaction-energy $E_i$ according to (9.10), (□) self-energy $E_0$ according to (9.14). (The curves have been rescaled such that their maximum equals 1.) $t$ is time.

reconnection has to be found elsewhere. This will be further discussed in Chapter 11.

Though the experiment mentioned above has been relatively simple, we have been able to compare one quantitative result. For both the vorton rings and the smoke rings of low axis ratio ($L/2R_e \leq 3.5$) the product of the period of one oscillation $\tau$ and the average ring velocity $V$ have been calculated [6]. In fig.10.13 the results are compared. For the vorton simulations both the N- and N+K-equation have been used. In both cases the number of vortons $N$ has been chosen such that the numerical results agreed best with the experimental results. This

[6]It can be shown that this quantity is independent of the circulation $\Gamma$. This is a fortunate circumstance, since $\Gamma$ is hard to measure. See [44] for details.



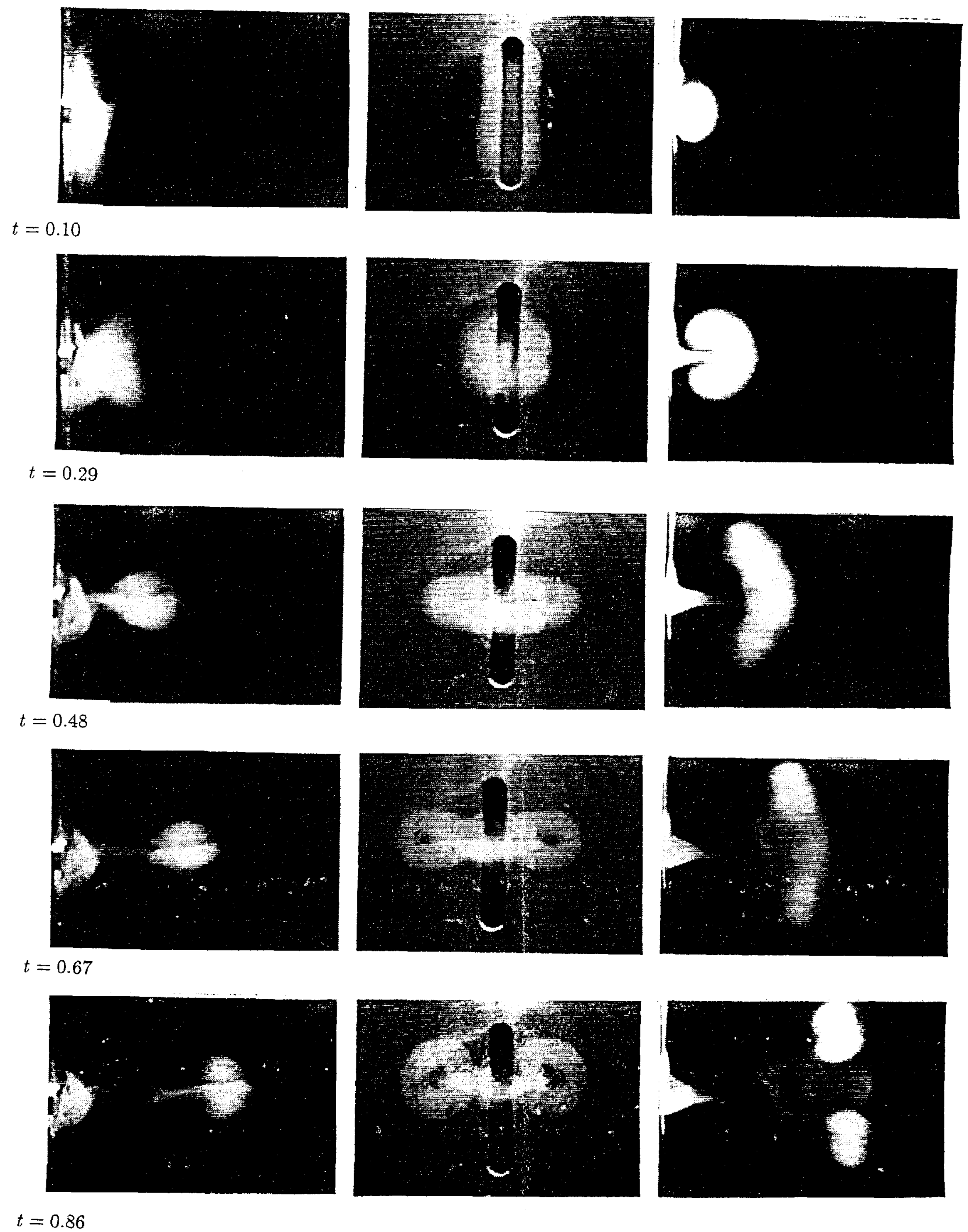




Figure 10.11

resulted in $N = 10$ in case of the N-equation and $N = 8$ in case of the N+K-equation. We observe agreement between numerical and experimental results with regard to the trend of the curves. The N-equation seems to perform somewhat better. However, the fact that vorton rings can only contain discrete values of $N$ strongly restricts the adaptation of the numerical ring to the experimental ring.

## 10.3 Head-on Collision of Two Coaxial Vorton Rings

### 10.3.1 Introduction

As remarked in Chapter 2, in 1858 Helmholtz had already showed that if two coaxial and identical vortex rings approach each other, their radii increase and their velocities decrease. He had also remarked that this situation could be compared to a single ring approaching a flat wall perpendicularly, in which case the ring collides with its own "mirror" image (compare fig.7.1). The former case will be called ring/ring interaction and the latter ring/wall interaction.

The initial configuration of ring/ring interaction is shown in fig.10.14. Notice that this configuration can only represent ring/wall interaction if the free-slip condition on the wall is valid, i.e. if no condition is imposed on the tangential velocity at the (imaginary) wall.

In §A.2 of the Interlude we mentioned Dyson's impressive analytical results of 1893 on this configuration. After Dyson's remarkable paper, it seems to have lasted more than 70 years before interest in the interaction of vortex structures with planes, or other objects, revived. Direct incentive for this revival can be found in the concern which arose in the early 1970s over the hazard presented by trailing vortices as produced by large aircrafts [276], which could interact with the ground and other aircrafts. One of the phenomena only then discovered was the rebound effect: at close approach towards a wall the movement of the vortex showed reversion, i.e. the vortex started to move away from the wall. Though initially the rebound was explained by inviscid core deformation (see [128]), it soon became clear that the effect was due to the influence of the boundary layer: when the distance between ring and wall is of the order of the apparent thickness of the core, the boundary layer is disturbed and a secondary ring is generated at the wall, which induces an upward motion on the approaching ring and se[illegible]s it from the wall (see e.g. [274]). After the rebound the first ring may again approach the wall since the secondary ring becomes weaker, and the rebound may occur again.

More generally, the rebound effect made clear that interaction of vortex structures with viscous boundaries (no-slip condition) are fundamentally different from inviscid boundary interactions (free-slip condition) represented by mirrored structures.

### 10.3.2 Recent Results from Literature

While results on ring/ring interaction are scarce, ring/wall interaction has been given more attention. Chu *et al.* [40] have done experimental work on rings approaching both a solid surface and a (slightly contaminated) free surface. For the circulation of the ring, they have found that during the free-travelling stage $\Gamma$ is almost constant, while during the vortex stretching at close approach of the surface $\Gamma$ strongly decreases. This violation of Kelvin's Circulation The-

Figure 10.11: (*see inserted sheet*) Single pseudo-elliptical vorton ring ($N = 8$, $L/2R_e = 7$): comparison between numerical simulation (in case of N+K-equation) and experimentally produced smoke ring. The vorton ring is represented by isosurfaces of $\mid \tilde{\mathbf{w}} \mid$ given by (9.18). Three views of the ring are given at each time $t$.

orem indicates that viscosity becomes influential at that stage. Apparently the only numerical simulation on this configuration has been performed by Kambe and co-workers. Their viscous vortex model also showed a variation of circulation: first it is constant, then it decreases at close approach. Energy appeared to decrease from the start [98].

The only recent experiment on head-on colliding vortex rings seems to have been performed by Lim *et al.* [128]. They have compared the ring/ring interaction with the ring/wall interaction. During the ring/ring interaction, the radius of each ring continues to increase, in good agreement with Dyson's equation (see fig.d of the Interlude). They have found no rebound effect for the ring/ring interaction and concluded that the generation of secondary vorticity at the wall is indeed the cause of rebound in the ring/wall case. For the ring/ring case, the experiments also showed that an azimuthal instability can develop along the rings. A remarkable consequence of this wave formation is the formation of smaller rings around the circumference of the original rings. Once formed, these move outward radially [7].

### 10.3.3 Vorton Simulations

We have simulated the head-on collision of two standard vorton rings with the number of vortons per ring $N = 36$ and the vortons in both rings located opposite each other. Initially, the rings are separated 4 times the initial radius $R$, i.e. $d = 4R$ and $R = 0.8\ cm$; see fig.10.14. Only the N+K-equation has been applied, since all three vorton deformation equations showed the same simulation results.

In fig.10.15 we compare the development of radius $R$ of the approaching rings as a function of their distance (i.e. the distance between opposite vortons), given by $d$. It shows good qualitative agreement with Dyson's analytical curve for the Kelvin-ring, given in fig.d of the Interlude [8]. The curve shows no rebound.

From fig.10.16 we derive that distance $d$ between the rings, defined as the distance between the opposite vortons in both rings, almost stops decreasing as radius $R$ of each ring starts to increase strongly. This moment will be called $t^*$ and it is about equal to 0.006. Comparison with fig.10.15 shows that $t^*$ is also the moment $R$ starts to increase severely.

Apparently, the rings have a core which hinders their approach beyond a (small) distance. It should be mentioned that Dyson's elaboration also predicts a lower limit for the distance $d$.

To find out what is happening to the core during collision, we show in fig.10.17 and fig.10.18 contour plots of $\mid \tilde{\boldsymbol{w}} \mid$ (given by (9.18)). The first figure shows the deformation of the core in plane $A$ of fig.10.14, the other the same result in plane $B$. In both figures, we depict at $t = 0$ the same maximum value of $\mid \tilde{\boldsymbol{w}} \mid$ in order to show the difference in the vorticity distribution between the two locations. At subsequent times, the values of $\mid \tilde{\boldsymbol{w}} \mid$ on the contour levels has been adapted to the overall maximum of $\mid \tilde{\boldsymbol{w}} \mid$ in each plane.

[7] This phenomenon resembles the formation of rings in the so-called Crow instability of rectilinear trailing vortices (see e.g. [276]).

[8] Quantitative comparison has not been attempted, since in that case one needs to know the core size $a$. As we have shown in §10.1.2, $a$ cannot be defined unambiguously.

Figure 10.12: (*see inserted sheet*) Single pseudo-elliptical vorton ring ($N = 96, L/2R_e = 7$): behaviour in case of the N+K-equation and a slight restriction (exaggeratedly illustrated by the sketch at bottom right). Tree views of the ring are given at each time $t$.

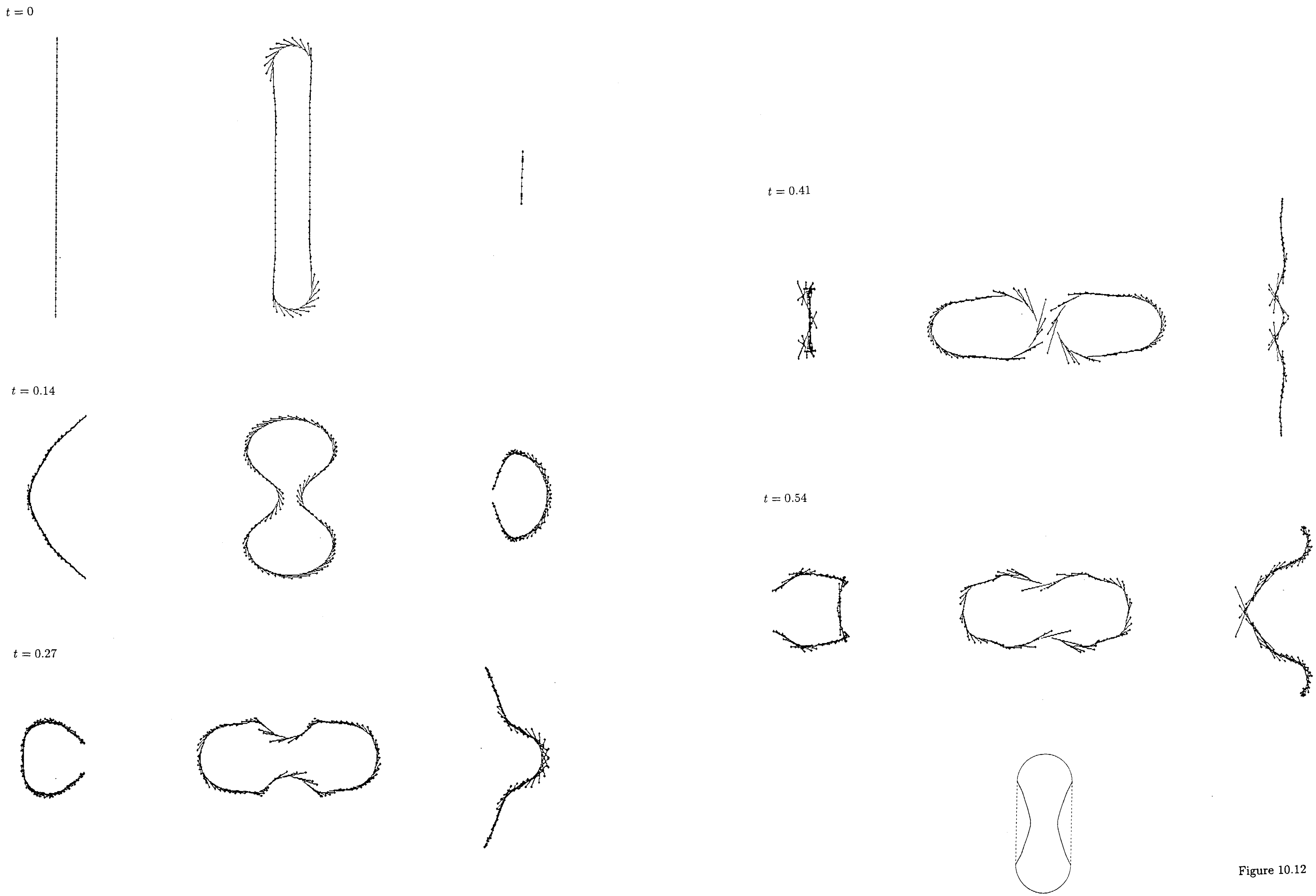



Figure 10.12

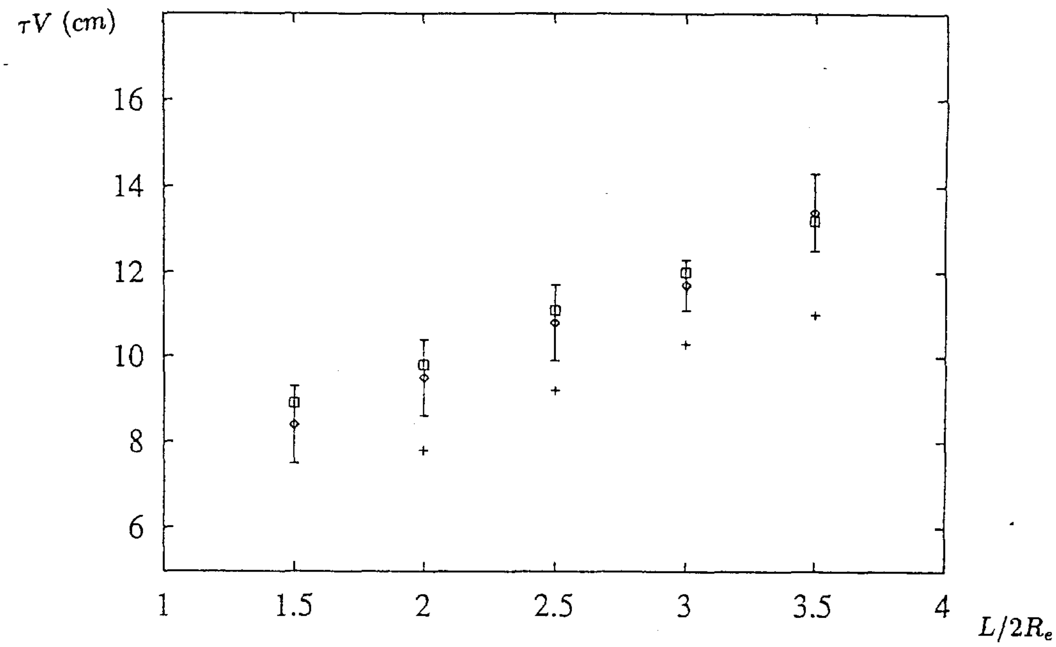



Figure 10.13: Single pseudo-elliptical vorton ring ($\Gamma = 225$): comparison of $\tau V$ ($\tau$ is the duration of one complete oscillation; $V$ is the average ring velocity) vs. axis ratio $L/2R_e$. Comparison between results from numerical simulations ((□) N-equation ($N = 10$) and (+) N+K-equation ($N = 8$)) and from experiment (◇) (error bars have been provided).

Comparing the time development of the cores in figs. 10.17 and 10.18 with figs. 10.15 and 10.16, we observe that at $t^*$, defined above, the cores become deformed and asymmetrical. This is also illustrated by fig.10.19(a) in which the development in time of the non-dimensional core radius $\tilde{a}$ (defined in 10.1) in plane $B$ (see fig.10.14) has been illustrated. The core is again derived from the velocity distribution around the centre line of the vorton ring (see fig.10.19(b)), but as is evident in the figure, the radius at the front of the ring (i.e. where the rings touch each other) finally becomes smaller, while the radius at the back starts to increase.

In fig.10.20 the development of circulation $\Gamma$ of each ring is presented as calculated from integration along the two different curves $A$ and $B$ indicated in fig.10.14. Comparing with figs. 10.15 and 10.16, we conclude that conservation of circulation starts to be violated at $t^*$. However, the direction of the deviation of $\Gamma$ depends on the curve used in the calculation. As mentioned in §10.3.2, decrease of $\Gamma$ has also been found in experimental and numerical experiments in literature, where it can be attributed to viscosity. For our simulation, this explanation is not workable. Apparently, the calculation of $\Gamma$ by means of the curves $A$ and $B$ is no longer allowed the moment the distance between the core centers has decreased beyond a certain value.

As the developments of $R$, $d$, and $\Gamma$ are now known, we can check Dyson's results for the rate of change of $R$, given in §A.2 of the Interlude. In fig.10.21 the relation between circulation $\Gamma$ divided by distance $d$ is plotted against the rate of change of radius $R$. We conclude that for

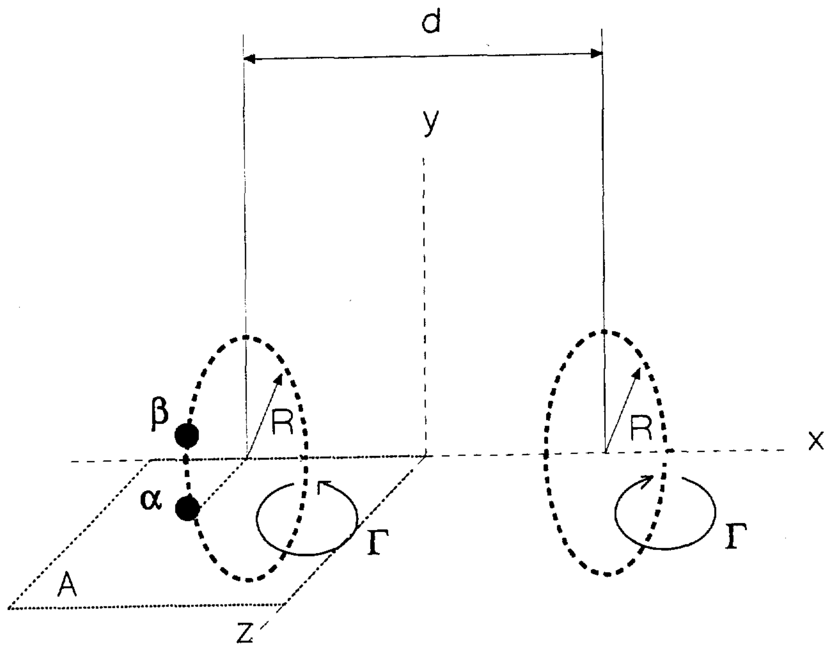



Figure 10.14: Initial configuration of two head-on colliding vortex rings (ring/ring interaction). The dots indicate vorton locations. Plane B (curve B) is plane A (curve A) rotated around the the $x$-axis, such that it intersects the vorton ring at the location between vortons $\alpha$ and $\beta$.

a certain period of time the linear relationship, as found by Dyson, exists. After $t^*$ the rings stop behaving as Kelvin-rings.

Regarding the conservation of the motion-invariants mentioned in §9.3.1, one may recognize that due to the symmetry of the configuration, expression (9.5) for linear momentum, expression (9.8) for angular momentum, and expression (9.16) for helicity are always zero. This is indeed confirmed by the simulation results, though some very small scattering around the zero line is present. In fig.10.22 the development of interaction-energy $E_i$, according to (9.10), and of self-energy $E_0$, according to (9.14), are shown. We observe that conservation of $E_i$ and $E_0$ is almost immediately violated. Furthermore, both curves show opposite trends.

In the second simulation of this section, we have tried to reproduce the phenomenon of the formation of smaller vortex rings on the circumference of the two initial rings, as observed by Lim [128] and discussed in §10.3.2. As the simulation presented above did not lead to sinusoidal disturbances by itself, we decided to introduce a slight axial sinusoidal disturbance to the rings. The simulation was done with standard rings consisting of $N = 72$ vortons each. The wave mode number was chosen $n = 12$ (compare (10.4)), and initial separation was $2R$. In fig.10.23 the simulation result is shown by means of the vorton locations. Initially, when the rings are separated at relatively large distances, the imposed wave mode is stable and the disturbance will be damped. However, when the rings approach each other, their velocities decrease and the wave mode becomes unstable at the moment when the non-dimensional velocity $\tilde{V}$ reaches the critical value for this specific mode (compare with fig.10.5). In the next phase, reconnection takes place and small rings are formed. Note, however, that these rings seem to be still connected to each other by vortons lying in between them.

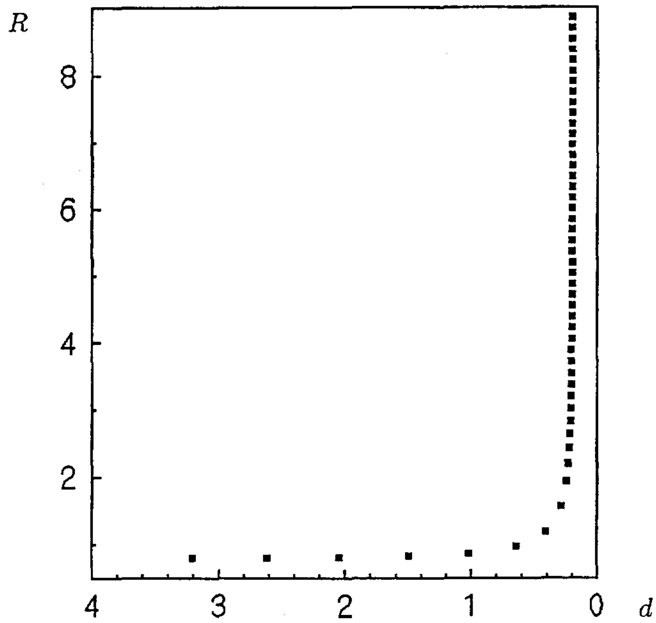



Figure 10.15: Head-on collision of two standard vorton rings ($N = 36$), initially separated $d = 4R$: radius $R$ vs. distance $d$.

Finally, we have investigated the effects of vorton division (see §9.4) on this configuration. At least one reason to apply vorton division in this case can be obtained from the simultation results presented above. In fig.10.18 we observe that the location of maximum vorticity, which is supposed to be the location of the core centre, starts to revert at a certain time, even though the vortons of both rings keep approaching each other as is shown by fig.10.16.

We have applied the vorton division procedure as described in §9.4. For $\lambda$ we have taken the value given by Pedrizzetti [177] from his analysis of the conservation of "vortex volume" to a vorton. In §10.1.1 we found that the core size of a vorton ring (for a circular, undeformed core) is proportional to the length of the vortex tube (see (10.2)). Supposing that the vortex volume is proportional to $a^2\Delta x$ ($\Delta x$ has been defined in fig.9.3), one can easily derive that $\lambda$ should be $2^{\frac{2}{3}}$. Notice that this value implies that stretching of a vortex tube will cause a decrease of its core size, whereas in the absence of division the core size grows during stretching.

We have applied vorton division to the case of two standard rings ($N = 36$) initially separated 4 times their radius. In fig.10.24 we show the development of interaction-energy $E_i$ (9.10) for three cases: no vorton division, division without updating of the vorton strengths in order to conserve circulation, and division with updating (see §9.4). We observe that our division procedure including updating is able to prevent the continuous decrease of the level of $E_i$ as happens without division. Division without updating is seen to give unsatisfactory results in this respect.

However, the simulation of this configuration has also made clear an important drawback of vorton division. As explained in §9.4, the added vortons are positioned at locations which are linearly interpolated between the locations of the existing vortons (see fig.9.3(b)). In case of rings this means a (slight) azimuthal disturbance on their shape. From our simulations we found that after division had occured several times, an azimuthal disturbance started to grow

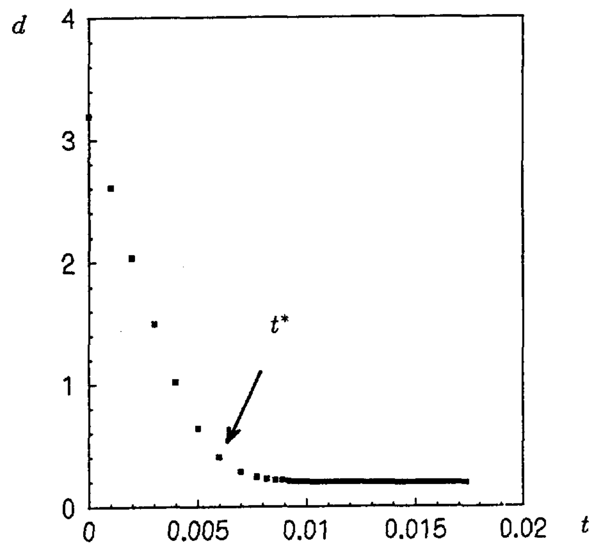



Figure 10.16: Head-on collision of two standard vorton rings ($N = 36$) initially separated $d = 4R$: distance $d$ vs. time $t$.

on the rings and the configuration became unstable. The wave number of this unstable mode appeared equal to the initial number of vortons, from which we conclude that the instability was not due to physical effects.

## 10.4 Oblique Interaction of Two Vorton Rings

### 10.4.1 Introduction

In §A.1 of the Interlude the experimental observation of the reconnection and subsequent oscillation of two obliquely inclined vortex rings by Wood [285] (1901) and by Northrup [166] (1911) were presented. The configuration they studied is in essence the one sketched in fig.10.25.

This configuration is not only challenging for experimentalists and for testing numerical codes, but is also of considerable interest as it is one of the most elementary configurations in which interaction and reconnection of vortex tubes can be studied in isolation, i.e. without other disturbing influences [9]. As already indicated in fig.e of the Interlude, this reconnection is generally supposed to be a competition between vortex stretching and smoothing by viscous stresses. The adjacent edges of the rings undergo severe strain and axial flow arises along the cores. At the same time, severe core deformation takes place.

Obviously, this process is very complicated. Only numerical simulations have enabled fluid dynamicists to get some more insight into several details of the actual phenomena, but in our opinion a really complete picture of reconnection, and its understanding, is still lacking.

[9] Another useful configuration in this respect may be the (pseudo-)elliptical vortex ring. However, in §10.2 we have seen that pseudo-elliptical vorton rings do not show the expected reconnection behaviour.

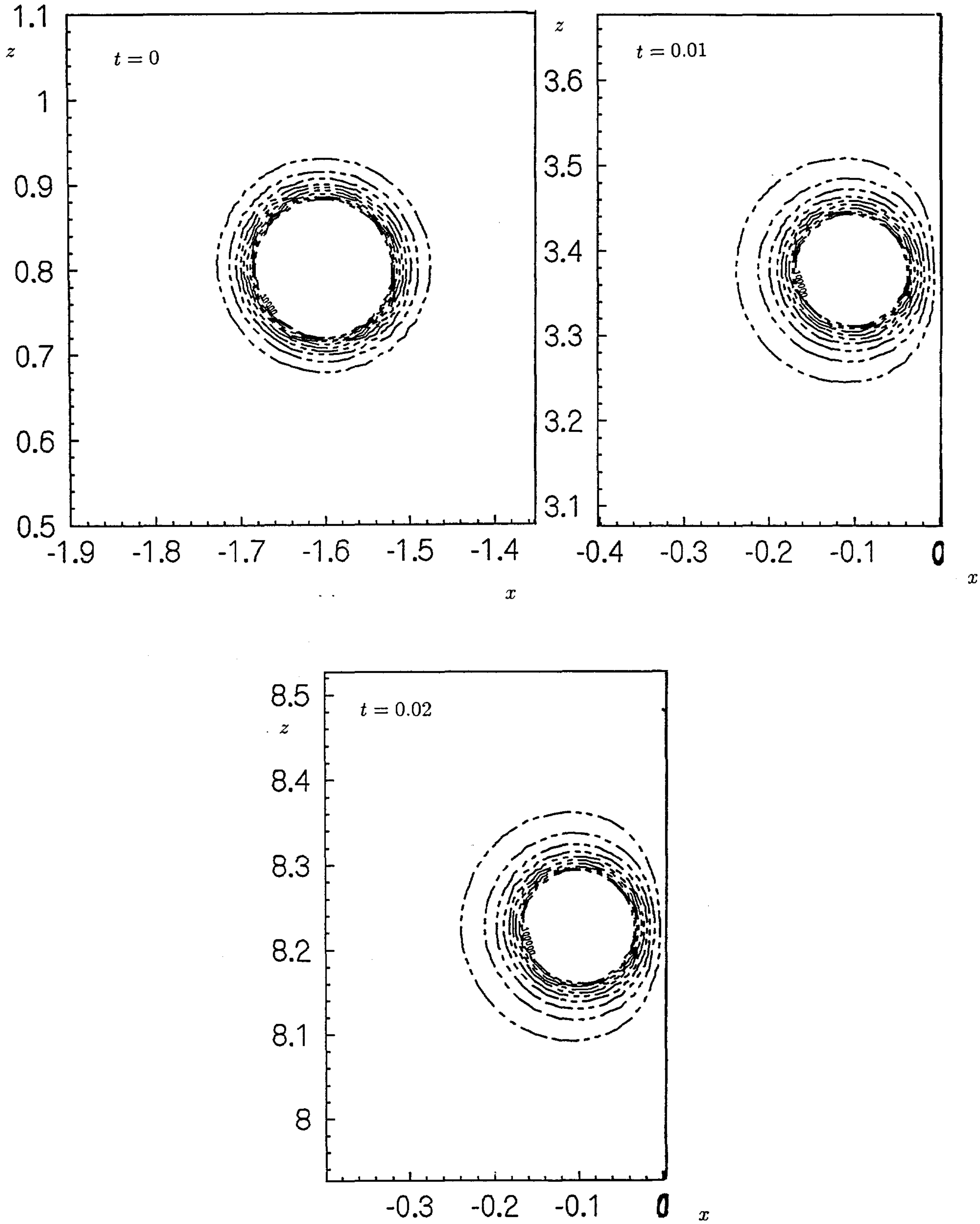



Figure 10.17: Head-on collision of two standard vorton rings ($N = 36$) initially separated $d = 4R$: deformation of the cores by means of contour lines of $| \bar{\mathbf{w}} |$ in plane $A$ (see fig.10.14). $t$ is time. Only the cross-section of the left ring is shown.

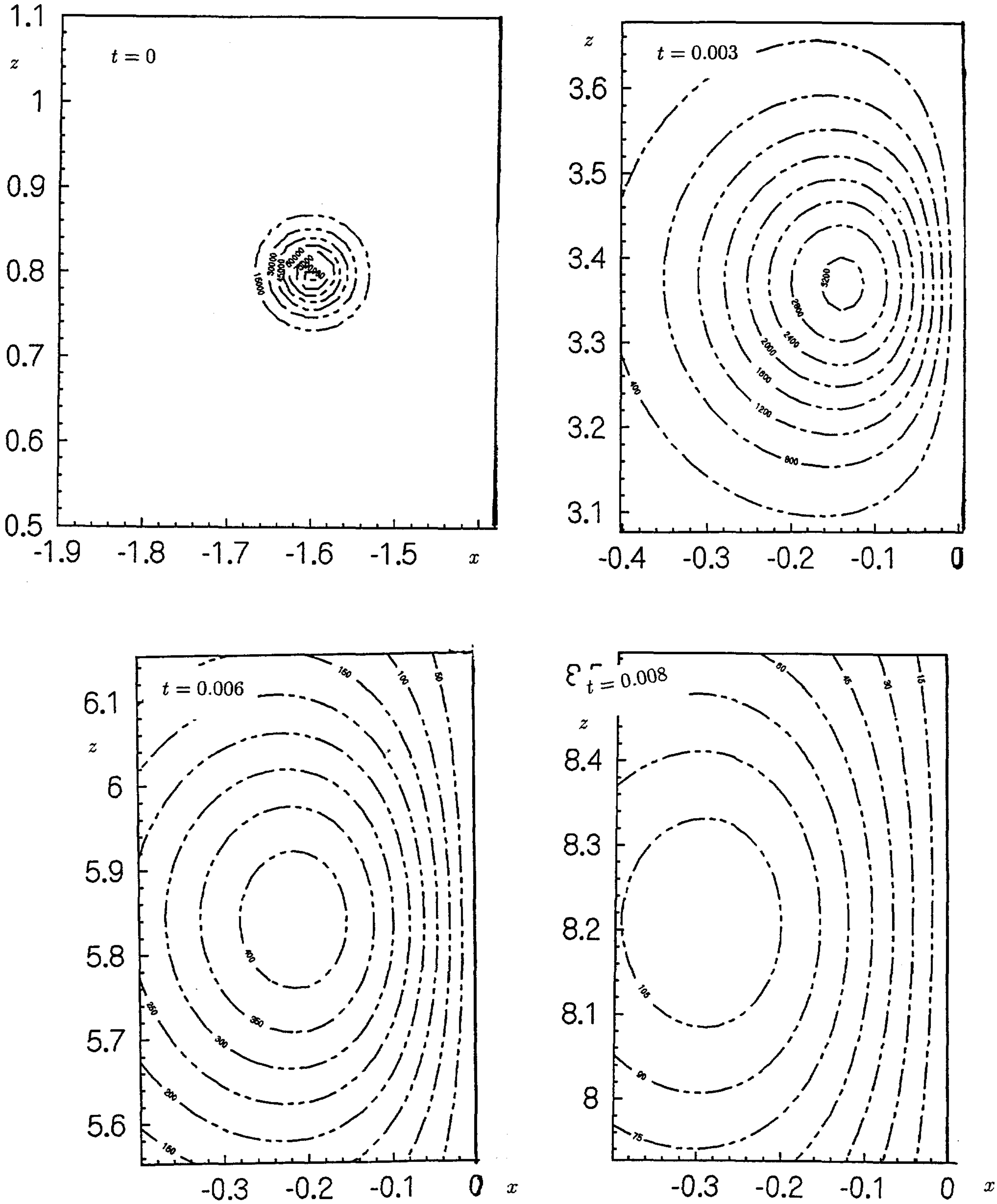



Figure 10.18: As fig.10.17, for plane $B$.

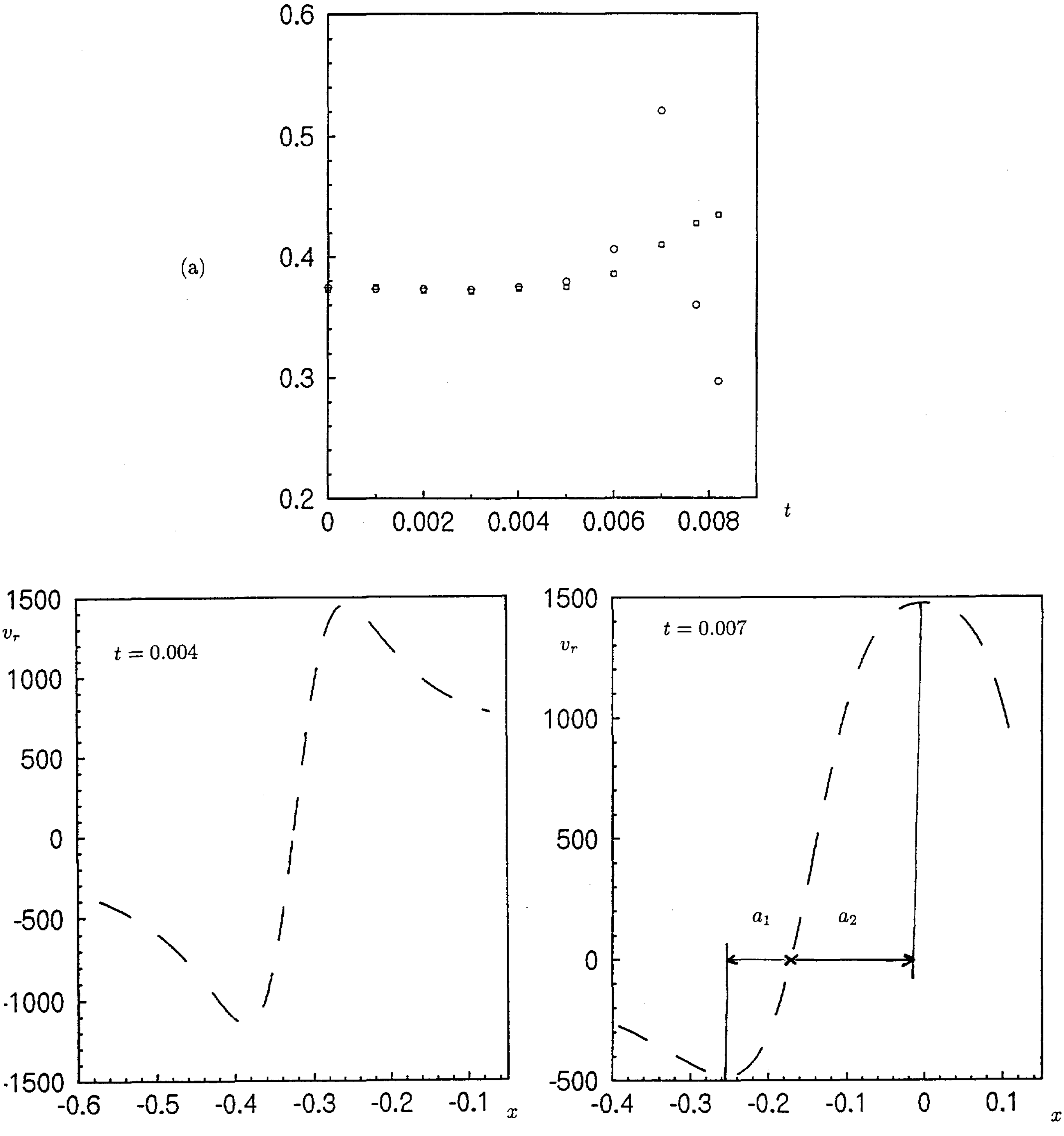



Figure 10.19: Head-on collision of two standard vorton rings ($N = 36$) initially separated $d = 4R$: (a) development of non-dimensional core radius (□) $\tilde{a}_1$ and (o) $\tilde{a}_2$ as derived from (b) the velocity distribution in plane $B$ (see fig.10.14) (*left*: before and *right*: after core deformation). Compare fig.10.2. $t$ is time.

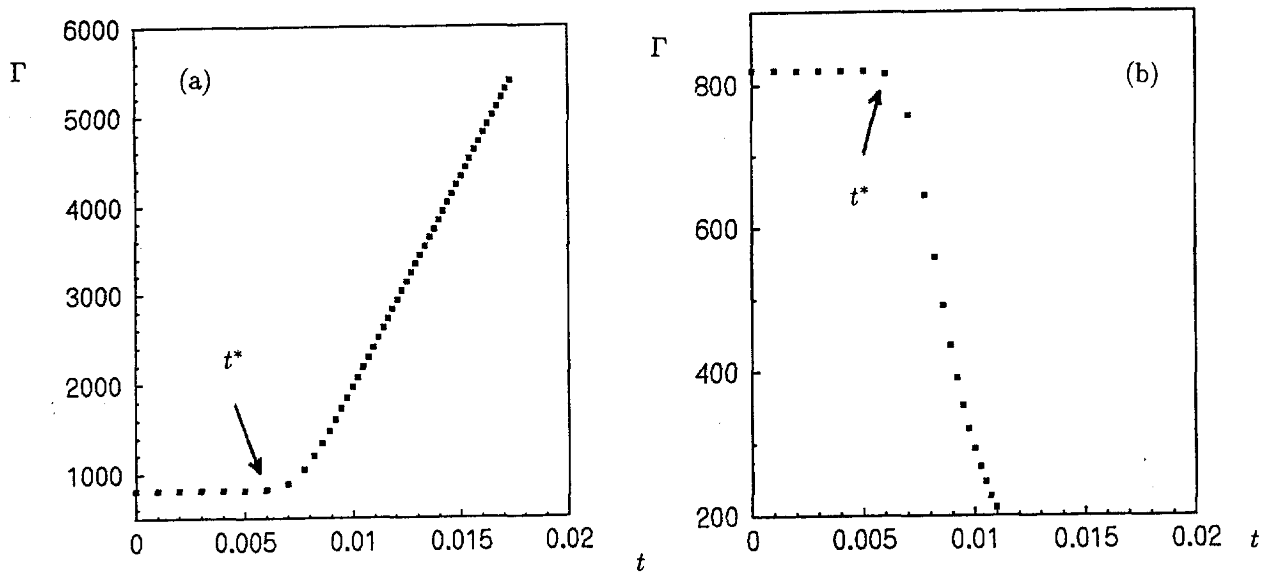



Figure 10.20: Head-on collision of two standard vorton rings ($N = 36$) initially separated $d = 4R$: circulation $\Gamma$ vs. time $t$: $\Gamma$ is calculated along (a) curve $A$ and (b) curve $B$ (see fig.10.14).

### 10.4.2 Recent Results from Literature

Only more than 60 years after Northrup, experimental work on the oblique interaction of vortex rings was taken up again, both in water and air. The experiments by Fohl & Turner and Oshima & Asaka (see [102] for references) have shown that the initial Reynolds number ($\mathrm{Re} = \Gamma/\nu$) and the angle $\theta$ of the rings with the horizontal (see fig.10.26) determine the interaction process. The latter authors have shown that three regimes of interaction can be observed, depending on the value of Re; see fig.10.26. For Re = 230-300, the two rings reconnect and merge into one elliptically shaped ring; after that this ring remains oscillating (stage $A$ in fig.10.26). For Re = 300-420, after the reconnection (stage $A$) the elliptical ring eventually splits up again into two rings (stage $B$). For still higher values of Re, the two rings, formed after splitting, reconnect again (stage $C$).

Recently, extensive experiments on this vortex ring interaction have been performed by Schatzle [208] and Izutsu & Oshima [93] (see also [171]).

The experiment by Izutsu & Oshima (IO) will be discussed here in some detail, since their results will be used to evaluate our vorton simulations. By means of hot wires they were able to measure the velocity field on a grid containing the two interacting vortex rings in air. From these measurements they calculated the vorticity field. In this way not only quantitative data could be obtained, but also the possible errors of interpretation have been avoided which are involved in the common method of visualizing vortex structures by means of tracer particles. Namely, the spatial pattern of these passive scalars does not faithfully represent the vorticity field, since a scalar only undergoes convection and does not undergo deformation. Therefore, at the locations of high vortex stretching, a depletion of tracer particles will occur.

In the IO experiment, the full formation of both rings took about 7.2 $ms$ after a loudspeaker was switched on to produce the rings. At that time, the radius of the vortex rings were

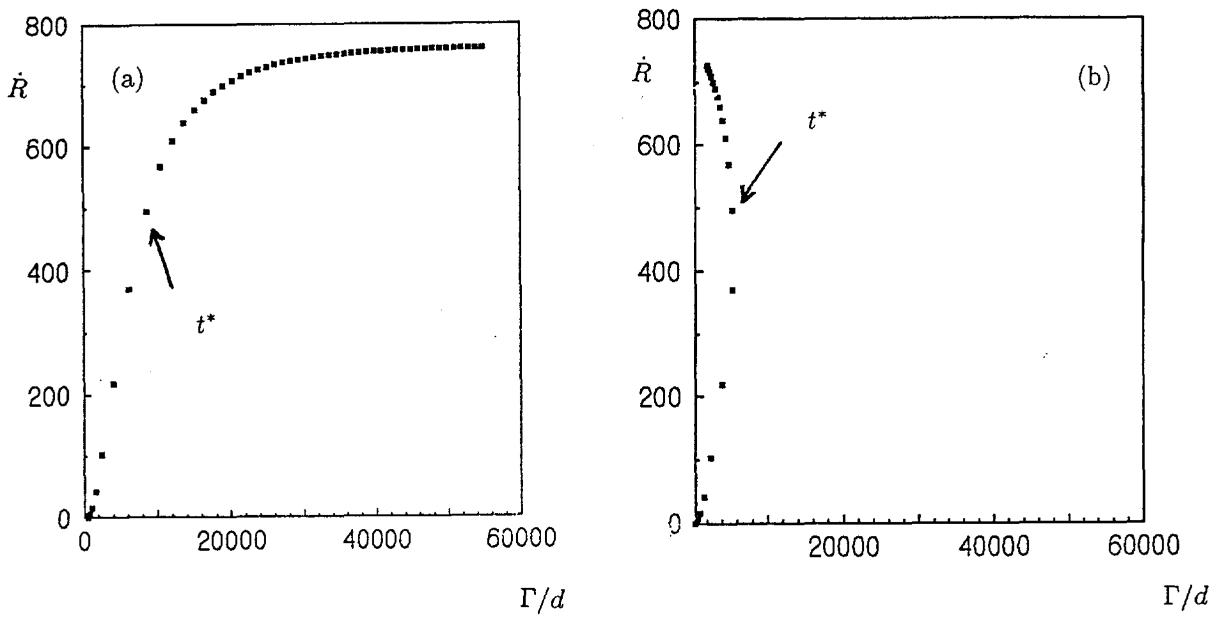



Figure 10.21: Head-on collision of two standard vorton rings ($N = 36$) initially separated $d = 4R$: $\Gamma/d$ vs. rate of change of ring radius, $\dot{R}$. $\Gamma$ is calculated along (a) curve $A$, (b) curve $B$ (see fig.10.14).

$R = 1.16R_0$ where $R_0 = 0.8$ $cm$ is the radius of the orifices. However, from their contour plots of vorticity, the radius appears to grow up to about $1.25R_0$ after the initial moment. The initial distance $D$ between the rings' centres was taken equal to $4R_0$. The initial circulation is given as $\Gamma_0 = 813.2$ $cm^2/s$ and the initial velocity $V_0$ of both rings as approximately 183 $cm/s$.

If we want to compare our numerical results with the experimental results presented by IO, we have to be sure our initial configuration resembles the initial configuration in the experiment as closely as possible. This means that we have to start the simulation with the same positions and characteristics of the rings. One check point for this correspondance is the angle of inclination $\theta$ of the rings during the early stage of the interaction. This angle is based on the locations of the core centers in the $x - y$ plane. IO mention $\theta = 86.3°$ at $t = 10$ $ms$ and $\theta = 81.3°$ at $t = 14.8$ $ms$ [10].

In fig.10.27 isosurfaces of vorticity magnitude are shown as derived by IO from their measurements. We observe that for this configuration, the interaction does not evolve beyond stage $A$ as shown in fig.10.26. Besides, the arrows indicate the presence of a weak pair of parallel vortex tubes, which have been called **threads** [11].

[10] These times are the times which have elapsed since the switching on of the loudspeaker in the experiment.

[11] In this thesis we will not speculate on the mechanism of reconnection which is exposed by these experiments or our numerical simulations. Comparison of our results with those by IO is only meant to investigate the applicability of the vorton method to this configuration. Nevertheless, we will mention shortly IO's conclusions on the reconnection process, as they have expressed it in [171]. According to IO, the actual cut-and-connect phenomenon does not take place at one location or moment, but "the cutting points dissolve gradually". The tubes of the approaching rings "consist of a number of vortex filaments ... Each filament individually cross-links one by one and moves away quickly in the direction normal to the plane of the filament because of its strong curvature". They suggest that an essential part is the bridging process, i.e. the formation of "new vorticity concentrations" which form the links of the two rings.

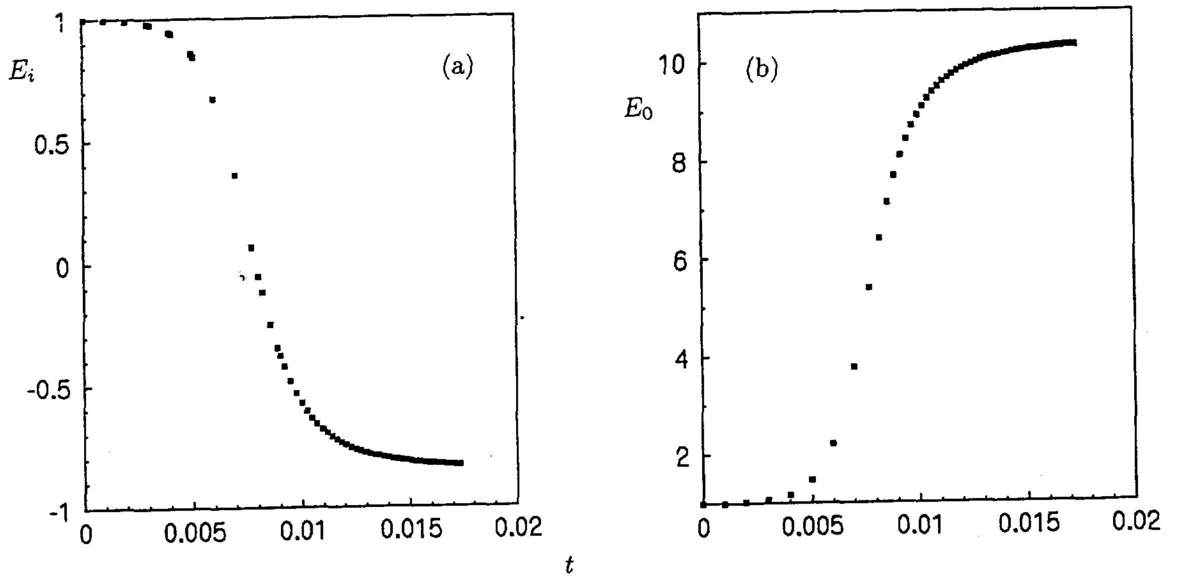



Figure 10.22: Head-on collision of two standard vorton rings ($N = 36$) initially separated $d = 4R$: the development (a) interaction-energy $E_i$ according to (9.10), (b) self-energy $E_0$ according to (9.14). Both curves have been scaled with their initial value. $t$ is time.

The configuration treated here has also been used for the investigation and calibration of several vortex methods and numerical methods. Besides, as remarked, numerical simulation is still the only way to investigate the reconnection process in more detail. Some examples of recent numerical research are presented:

- Anderson & Greengard in [9] have applied a vortex-filament method (see §7.3.1) and added a numerical scheme to simulate diffusion of vorticity. They found that the cores of the rings deformed but were not uniformly pressed against each other during the interaction at infinite Reynolds number. This, they suggest, means that the process depends on Re. They have also remarked that "perhaps the reconnection process is too subtle to admit representation by a universally valid model" and pointed at the impossibility of several Eulerian grid methods to represent the small scales that arise in the region of reconnection.

- Winckelmans [283] (also in [32]) applied both the vorton method and the soft-vorton method (see Appendix B) to a configuration in which the two rings were initially inclined at an angle $\theta = 75°$. The cores consisted of a symmetrical pattern of vorton rings centered around a central vorton ring; see fig.10.28. From his numerical results, Winckelmans concluded that the K-equation is preferable to the N-equation and N+K-equation. However, he added that the vorton method is not applicable to this configuration and cannot represent reconnection. The conservation of motion-invariants (linear momentum $\boldsymbol{P}$ as given by (9.5) and interaction-energy $E_i$ as given by (9.10)) appeared to be severely

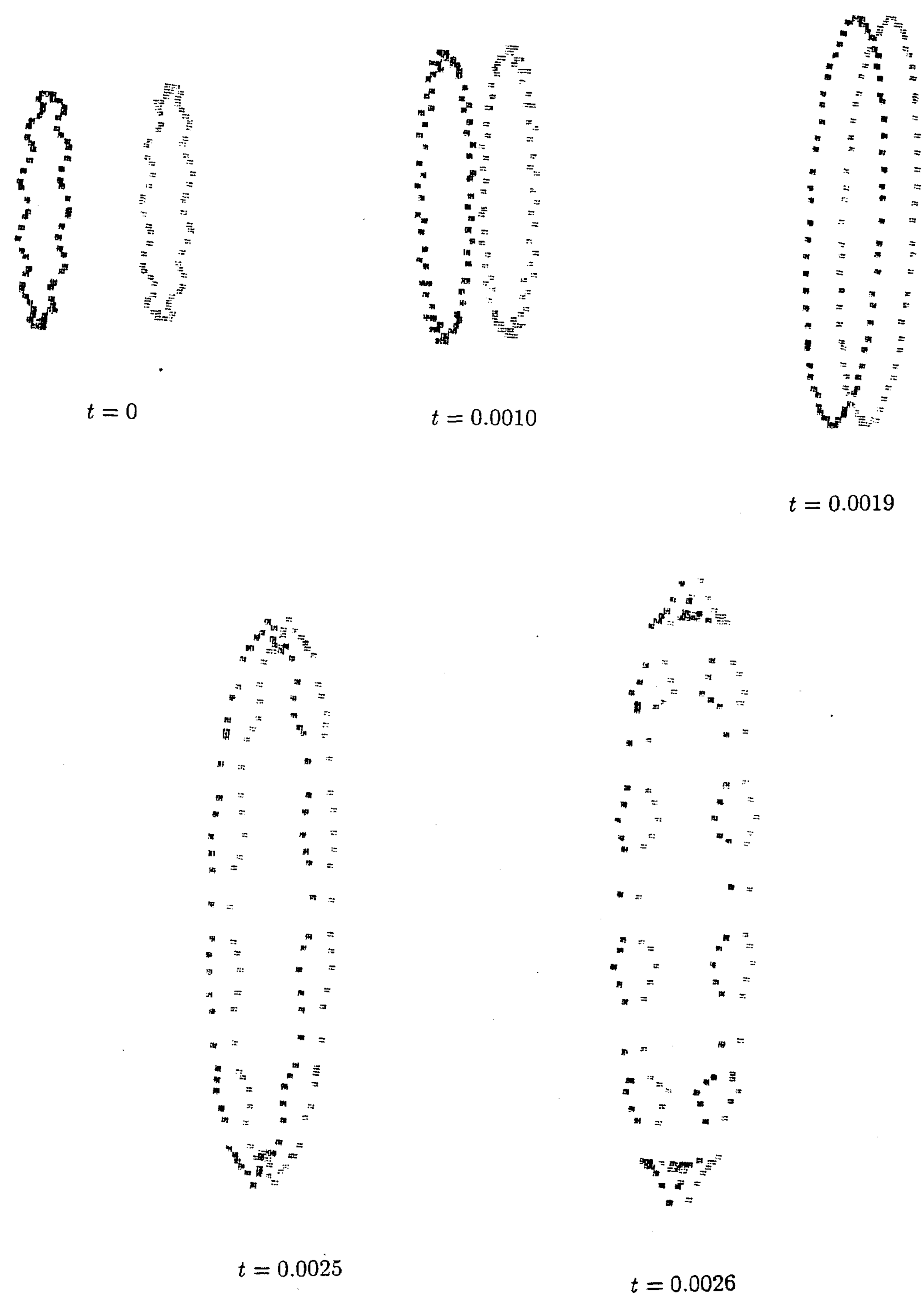



Figure 10.23: Head-on collision of two vorton rings ($\Gamma = 10$, $R = 2$, $N = 72$, initial distance $d = 2R$) with initial axial sinusoidal disturbance of wave mode number $n = 12$. Dots indicate vorton locations. $t$ is time.

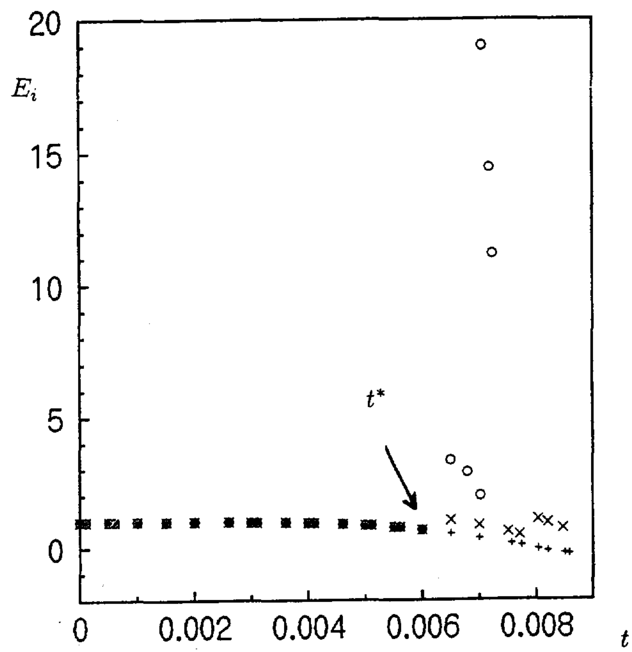



Figure 10.24: Effect of vorton division on the head-on collision of two standard vorton rings ($N = 36$) initially separated $d = 4R$: interaction-energy $E_i$ (scaled with initial value) vs. time $t$ in case of: (+) no vorton division, (o) vorton division without updating, (×) vorton division with updating.

violated at the moment of reconnection.

With regard to the soft-vorton method, he concluded that this vortex method *may* be useful, though no complete reconnection occurred in his simulations. He found none of the three vorton deformation equations to be preferable. The conservation of the motion-invariants mentioned was again violated. From the simulation of the same configuration by means of a soft-vorton method including a viscous diffusion term [12], Winckelmans concluded that the "physics of the problem" was well reproduced (he only investigated the K-equation). Linear momentum was not conserved, though, as he remarked himself, in unbounded viscous flows it should be. Winckelmans's simulations also showed threads.

- Kida *et al.* [103] performed extensive numerical studies of the reconnection of two viscous vortex rings, using several initial values for $\theta$, $R$, $D$, and also for viscosity $\nu$ and core radius $a$. The flow was simulated solving the Navier-Stokes equation by means of a spectral method. The simulations showed the process of formation of so-called bridges and threads, as introduced earlier by Hussain and co-workers in work on the reconnection of two anti-parallel vortex tubes with a sinusoidal disturbance (see e.g. Melander & Hussain in [160]). The reconnection showed three stages (see fig.10.29 and compare with fig.e of the Interlude):

  1. core deformation and stretching during collision of the closest parts of the rings

[12] This viscous vorton scheme will not be explained in this thesis. For details we refer to Winckelmans's thesis [283].

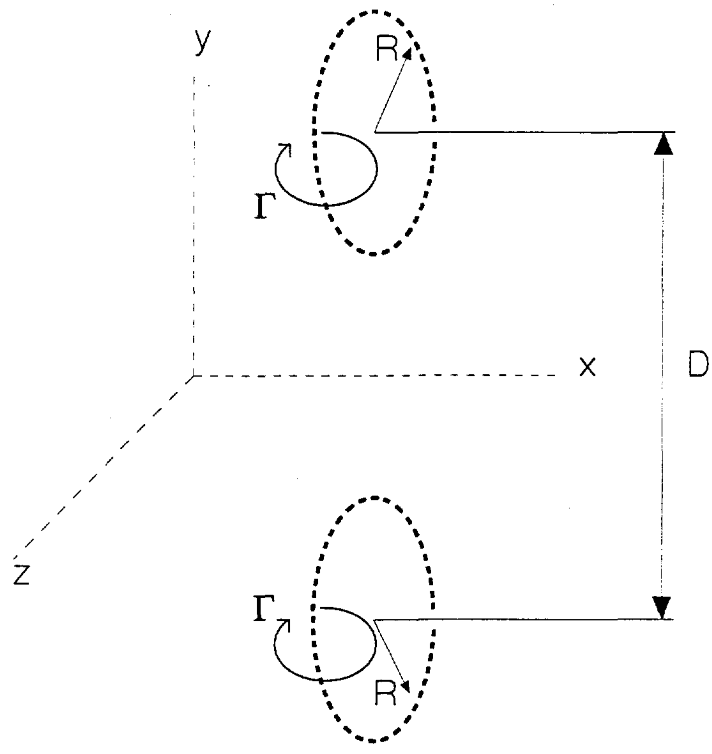



Figure 10.25: Initial configuration of the oblique interaction of two vortex rings.

(figures (a) and (b));

2. annihilation of vorticity due to diffusion and bridging, causing dramatic change in topology due to cross-linking of vortex-lines; bridging, or cross-linking, is the formation of connections between the two original rings, while unlinking is the process of annihilation of the colliding anti-parallel parts of the original vortex rings (figures (c) and (d));
3. threading, during which a remnant (unreconnected part) of the original vortex pair is sustained by stretching of the newly formed bridges (figures (d) and (e); see also the arrows in fig.10.27).

The authors also discussed the twisting of vortex-lines during reconnection. They argued that reconnection cannot occur in inviscid flows since "both topology and circulation of vortex lines do not change in time" [103, p.584].

For comparison with some of our numerical results, we show in fig.10.30 the contours of the vorticity magnitude in one of the symmetry-planes of the configuration, i.e. the $x-y$-plane of fig.10.25.

- Aref & Zawadzki (in [160]; see also [15]) used a vortex-in-cell method (see §7.3.2), with which they could simulate a slightly viscous flow. Their simulations showed weak threads during reconnection, but, unlike the results found by Kida *et al.* (see fig.10.30), these soon disappeared completely.

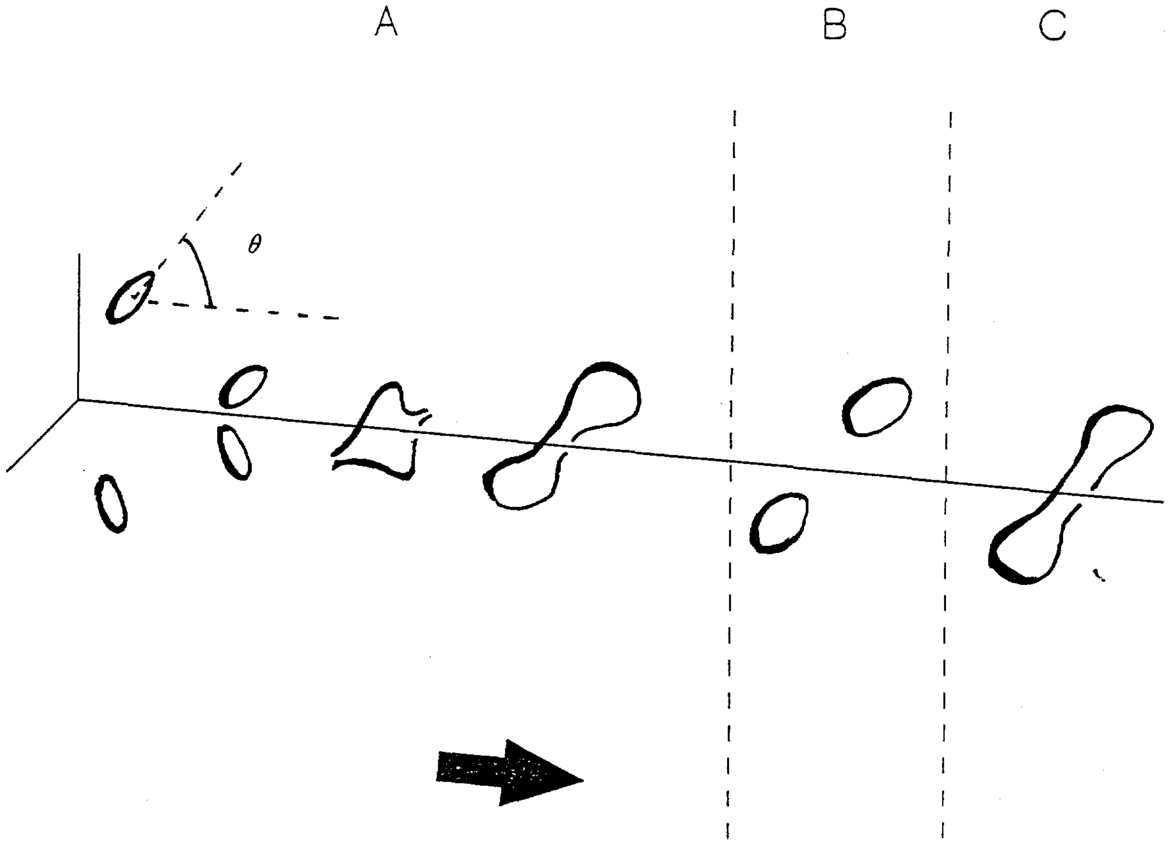



Figure 10.26: Three possible, consecutive, stages ($A$, $B$, $C$) in the development of the oblique interaction of two vortex rings with initial configuration as in fig.10.25 (see text). The arrow indicates time development. Compare fig.10.27.

### 10.4.3 Vorton Simulations

As mentioned, the numerical simulations presented in this section are compared with the experiment by Izutsu & Oshima (IO) [93]. For the initial configuration of fig.10.25 we have taken: $R = 1.25R_0 = 1.0\ cm$; $D = 4R_0 = 3.2\ cm$; $\Gamma = 820\ cm^2/s$; and for the number of vortons $N = 15$ (based on the initial ring velocity $V = 183\ cm/s$).

One has to realize that the initial positions of the vortons in the rings may be influential on the results. Initially, we will study the two configurations shown in fig.10.31. We can indicate these by the value of the ratio between the angle $\phi$ as indicated in fig.10.31(b) and $2\pi/N$, the initial angle between the vortons in the ring.

The simulations of both configurations have been performed for the N-, the K-, and the N+K-equation. Therefore, we have six possible combinations to investigate. In fig.10.32, the simulation results are shown by means of the vortons in the rings.

Above we mentioned the possibility to check the agreement between our numerical and IO's experimental initial condition. In all six cases of our simulation, the angle of inclination $\theta$ of 86.3° was reached at $t \approx 18.5\ ms$ (IO: 10 $ms$) and that of 81.3° at $t \approx 33.5\ ms$ (IO: 14.8 $ms$). This means that a certain amount of time has to be subtracted from the timesteps mentioned in fig.10.32 in order to get a proper comparison with the experimental results.

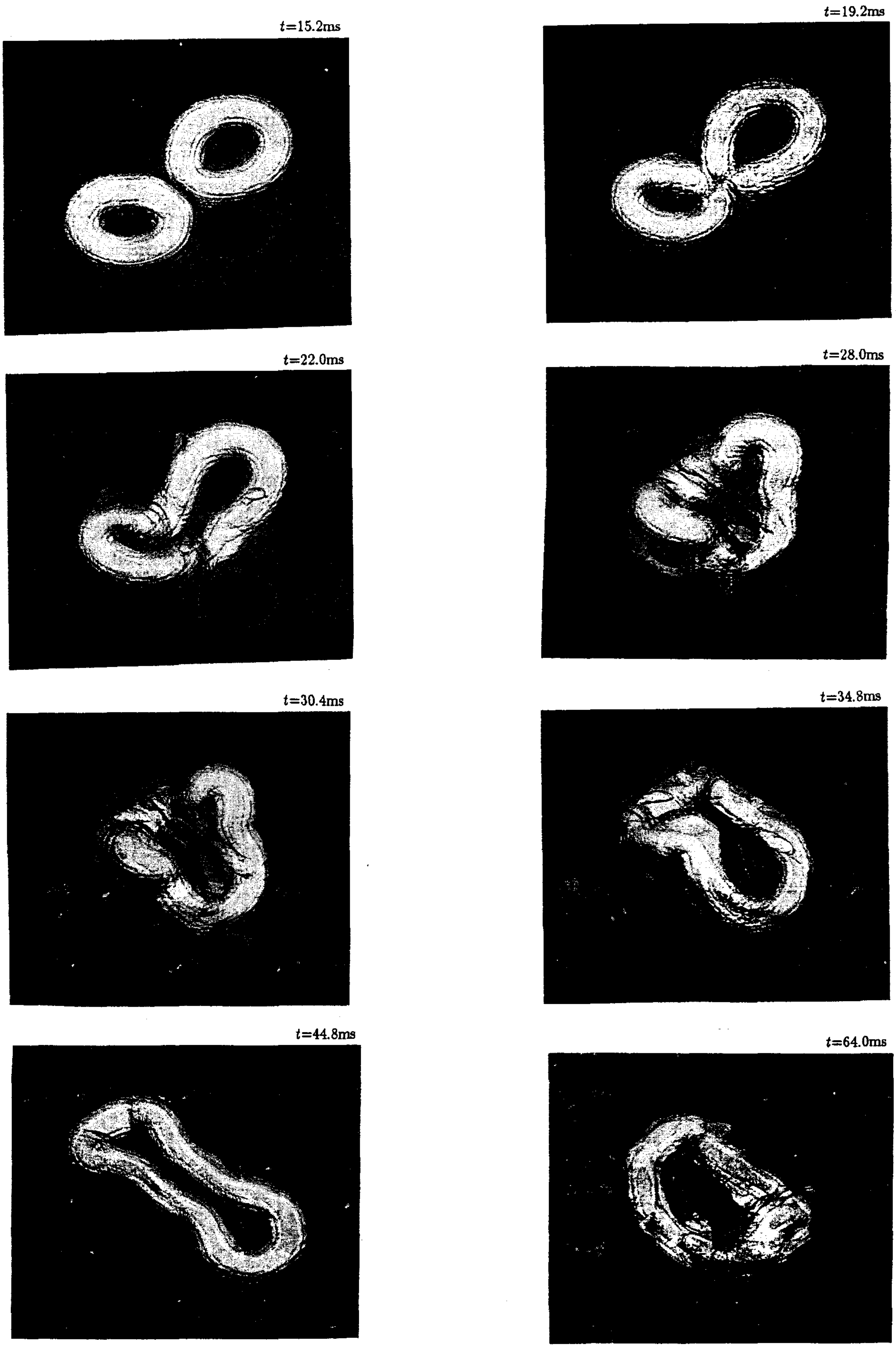



Figure 10.27: Isosurfaces of the vorticity magnitude for oblique interaction of two vortex rings. $t$ is time. Arrow indicates threads. Experimental results from Izutsu & Oshima [93].

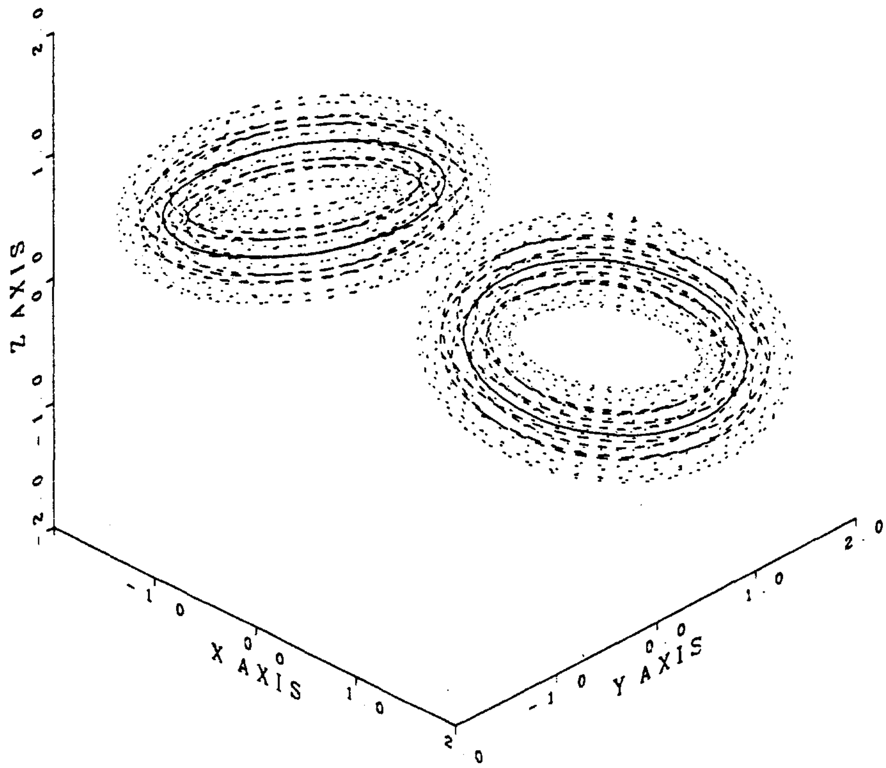



Figure 10.28: Initial configuration of a numerical simulation performed by Winckelmans. From [283].

However, we also have to conclude in our simulations the development of the configuration is slower. One possible explanation can be found in a disagreement between IO's and our definition of the inclination angle $\theta$ or of the ring velocity $V$. This result warns us that a quantitative comparison will be without meaning. However, below it will become clear that qualitative comparison makes sense.

From fig.10.32 we observe that the simulations of the configuration according to fig.10.31(a) show the formation of a "dipole" consisting of two anti-parallel vortons which eventually move away in a direction opposite to the movement of the reconnected vorton rings [13]. This dipole-structure does not seem to have occurred in the IO experiment. At this moment, it is unclear whether it has any relation with the threads mentioned above [14].

Though all simulations for case (i) in fig.10.32 do not agree with the IO experimental results, we observe that the alignment of the vortons is better conserved in the case of the N+K-equation than in case of the N-equation.

For the configuration of fig.10.31(b) we observe that the simulation of fig.10.32(b)(ii) (i.e. the K-equation) blew up after a short time. If we compare the other simulations with the results presented by IO as shown in fig.10.27, we observe that only the simulation of fig.10.32(c)(ii)

[13]For this reason, the "dipoles" disappear out of the pictures in fig.10.32. In passing, we have to remark that this dipole "blew up" (i.e. the strengths of both vortons grew indefinitely) when the vorton equations were completely solved for *all* vortons, while in the simulation of fig.10.32 advantage was taken of the symmetry of the configuration, i.e. only for one ring the equations were solved after which the new configuration was mirrored. Apparently, the behaviour of these dipoles depends strongly on the accuracy of the numerical procedure.

[14]Takaki & Hussain (see [13]) have remarked that two curved vortex tubes or ring before and after reconnection differ by a vortex ring.

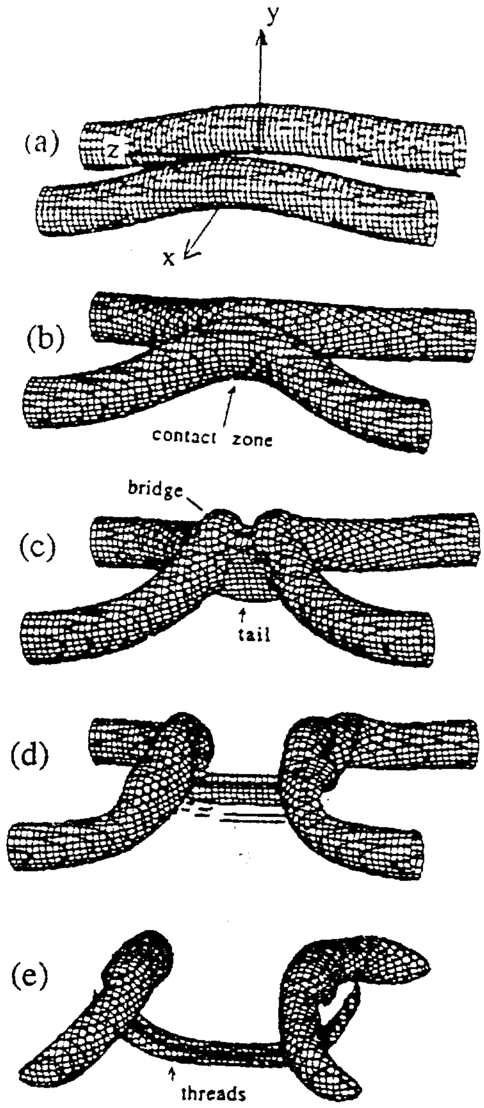



Figure 10.29: The reconnection process as proposed by Melander & Hussain in [160] (see text). Time development is from top to bottom.

(i.e. the N+K-equation) gives good agreement.

Further evidence of the better performance of the N+K-equation as compared to the N-equation is obtained from comparison of the development of some of the motion-invariants introduced in §9.3.1; see fig.10.33. Again, we remark that interaction-energy $E_i$ and self-energy $E_0$ (not shown) show opposite time developments.

In the rest of this section, we will only present results related to the simulation presented in fig.10.32(c)(ii), i.e. configuration with $\phi/(2\pi/N) = 0.5$ and application of the N+K-equation. In fig.10.34 isosurfaces of $| \bar{\boldsymbol{w}} |$ (given by (9.18)) are shown. Comparing these results with those from the numerical simulation by Kida *et al.*, shown in fig.10.30, we observe that our simulations do not show the same core deformation as in their case (in which a "tail" is formed behind the core centres). Though fig.10.34(c) suggest the presence of a thread-like object, we

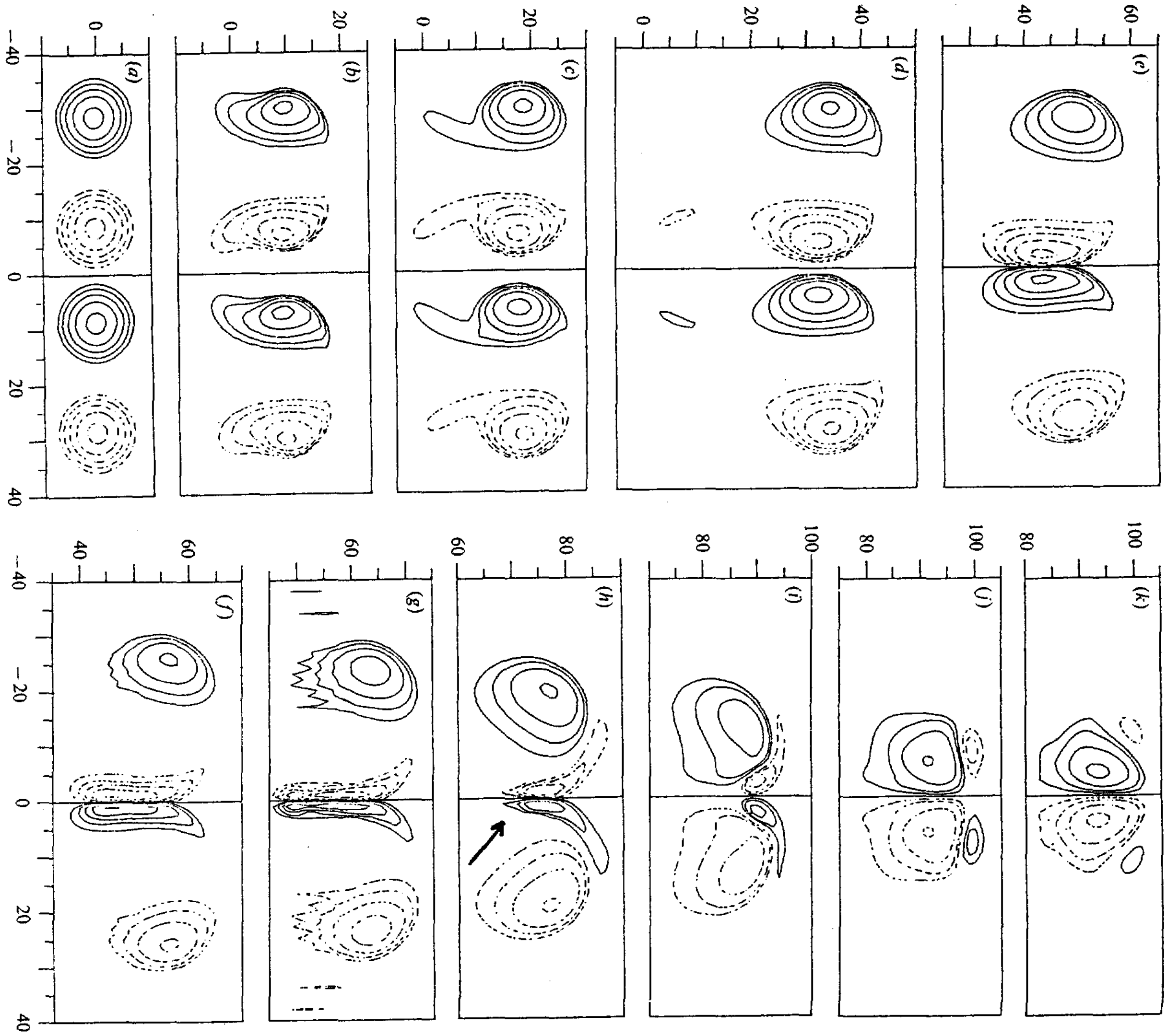


Figure 10.30: Contours of the vorticity magnitude from the numerical simulations by Kida *et al.* [103]. Cross-sections are in the $x-y$-plane (see fig.10.25). Time development is from left to right and from top to bottom. The arrow indicates the threads.

could not find a clear indication for this, as Kida *et al.* (see fig.10.30) and IO (see fig.10.27) did.

A final comparison of our numerical results and IO's experimental results (see fig.10.27) can be made by means of fig.10.35 with regard to the isosurfaces of vorticity magnitude. We observe that the development in time differs, as we had already expected from the comparison of the evolution of the inclination angle $\theta$, discussed above. Furthermore, fig.10.35(c) shows the absence of any thread-like structure in our simulated configuration. Finally, we have to remark that whereas IO's experiment seems to have ended in complete dissolution (due to viscosity) of the last configuration shown in fig.10.27, in our simulation the configuration shown in fig.10.35(f) continued to oscillate until it finally broke down into two unlinked vorton rings.

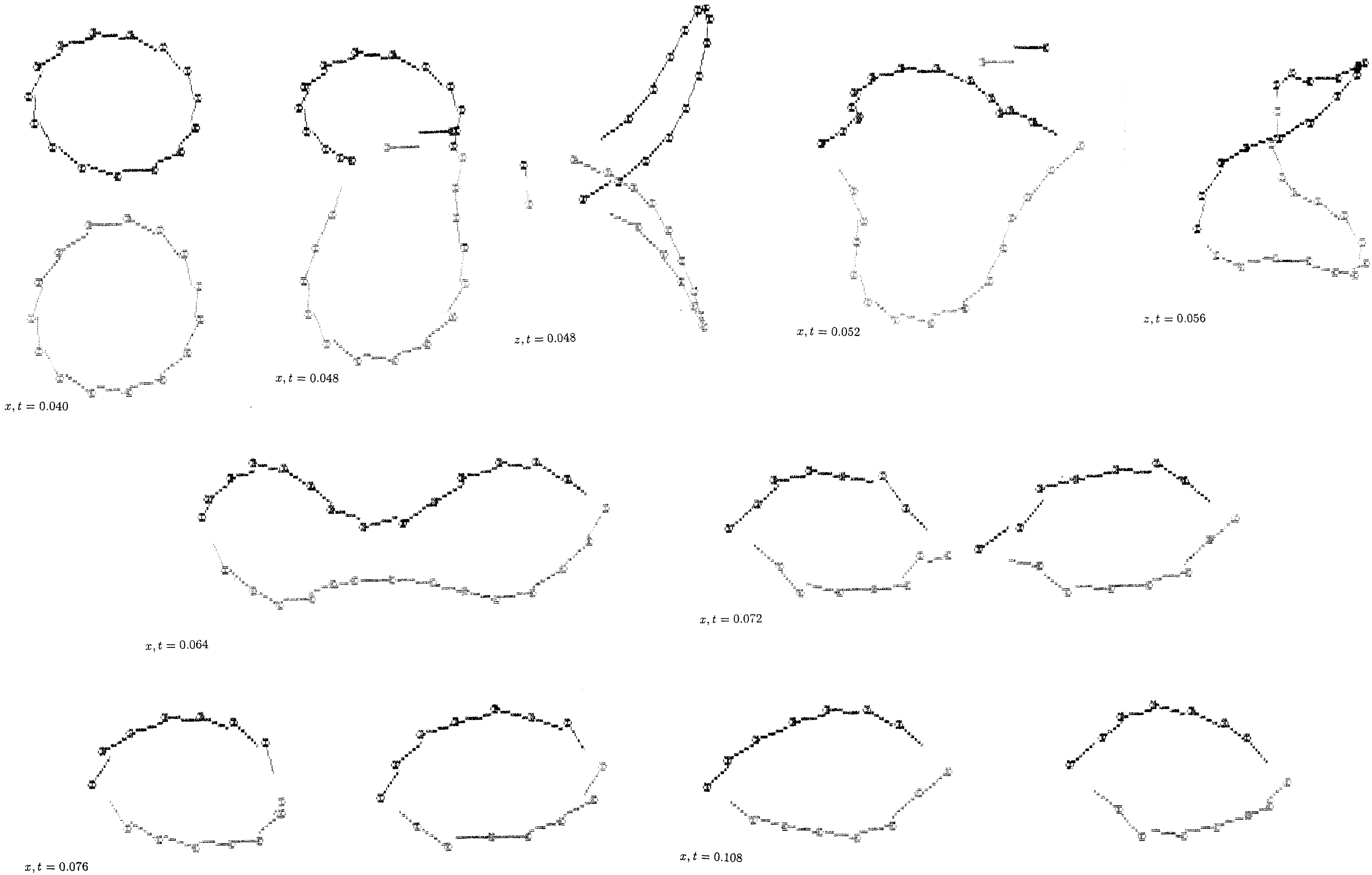



Figure 10.32 (a) (i)

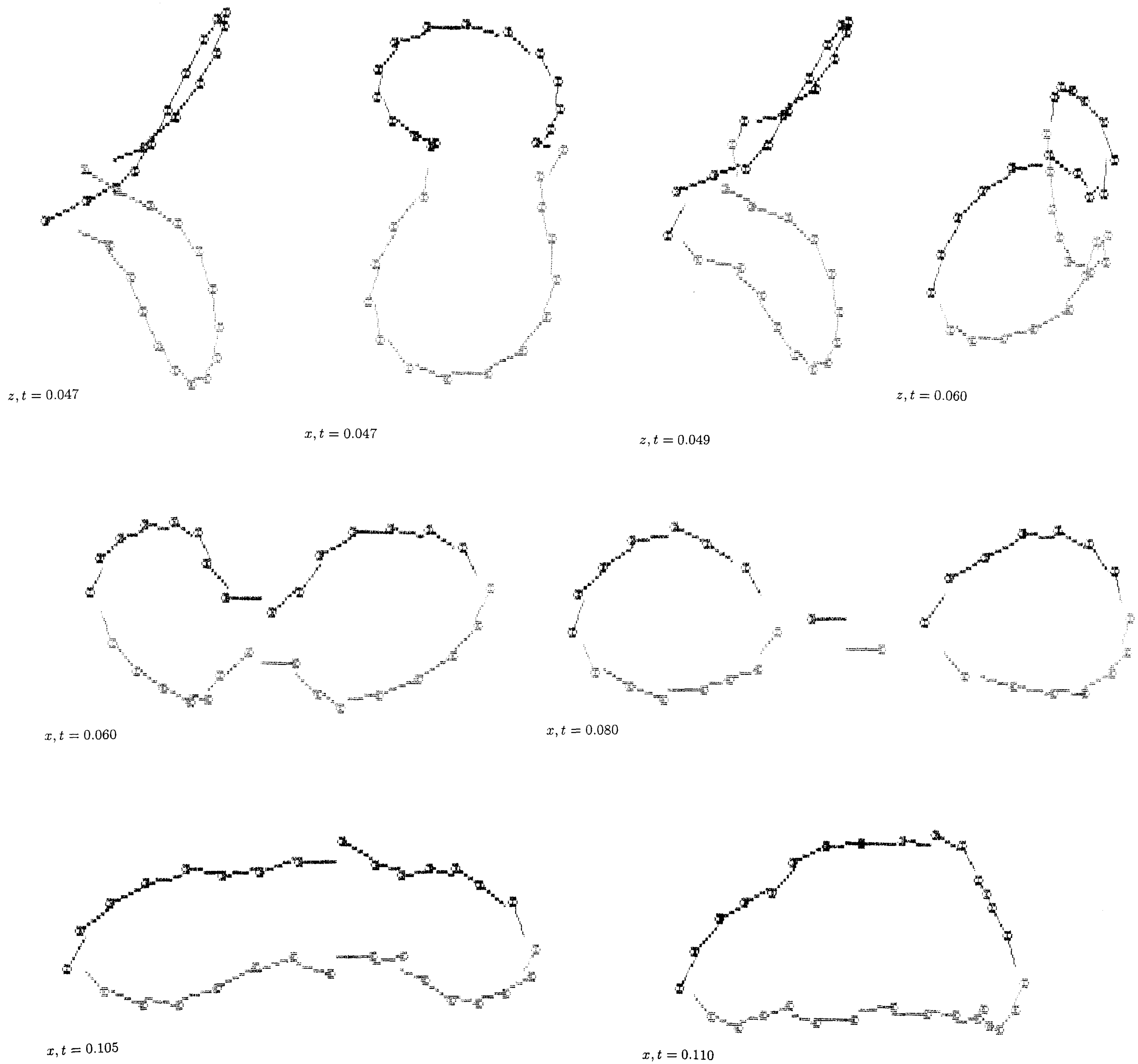



Figure 10.32 (a) (ii)

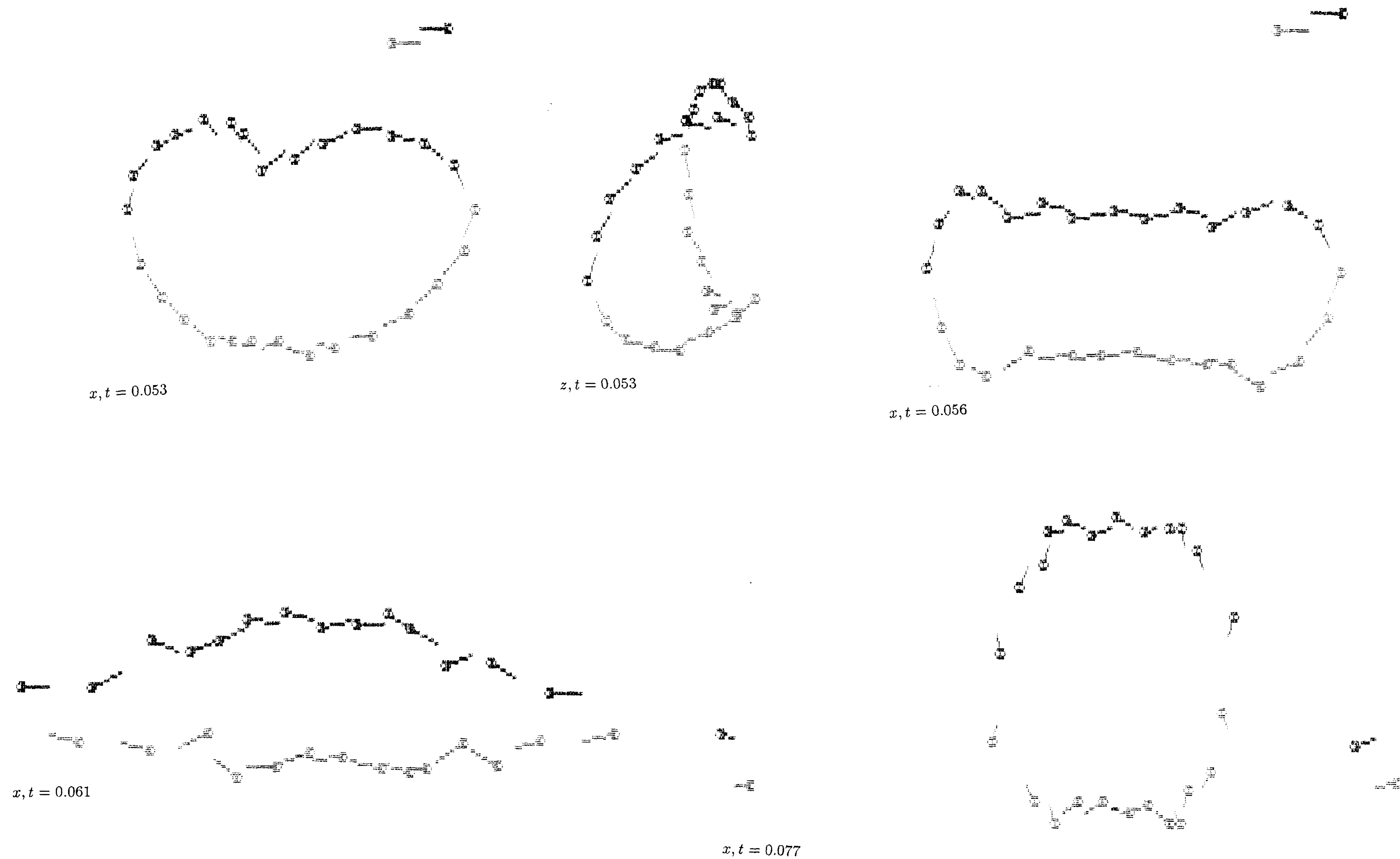
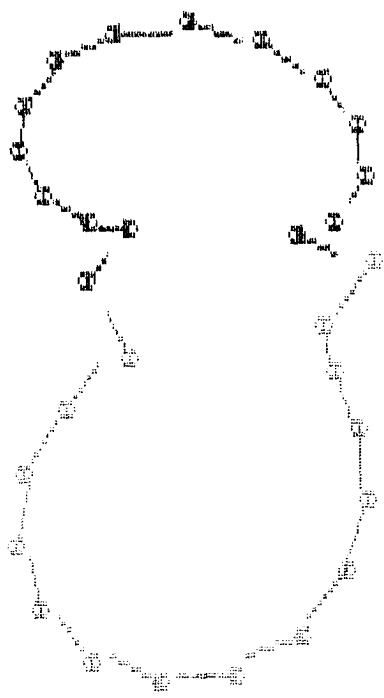
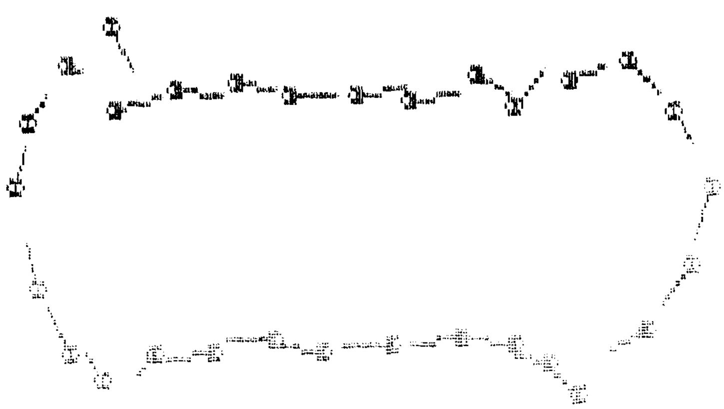


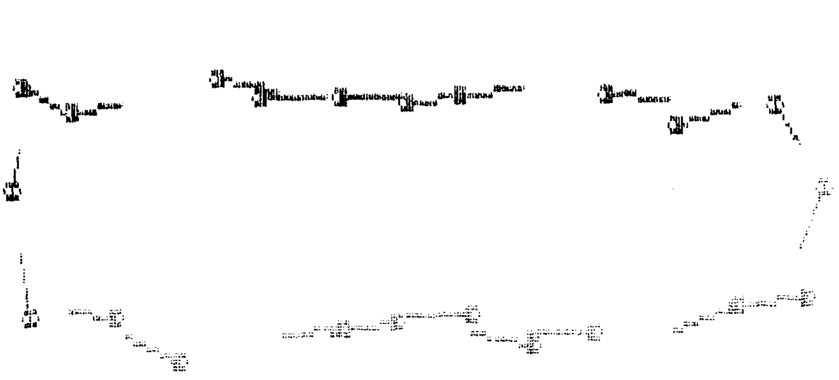

Figure 10.32 (b) (i)



$x, t = 0.047$



$x, t = 0.053$



$x, t = 0.057$

Figure 10.32 (b) (ii)

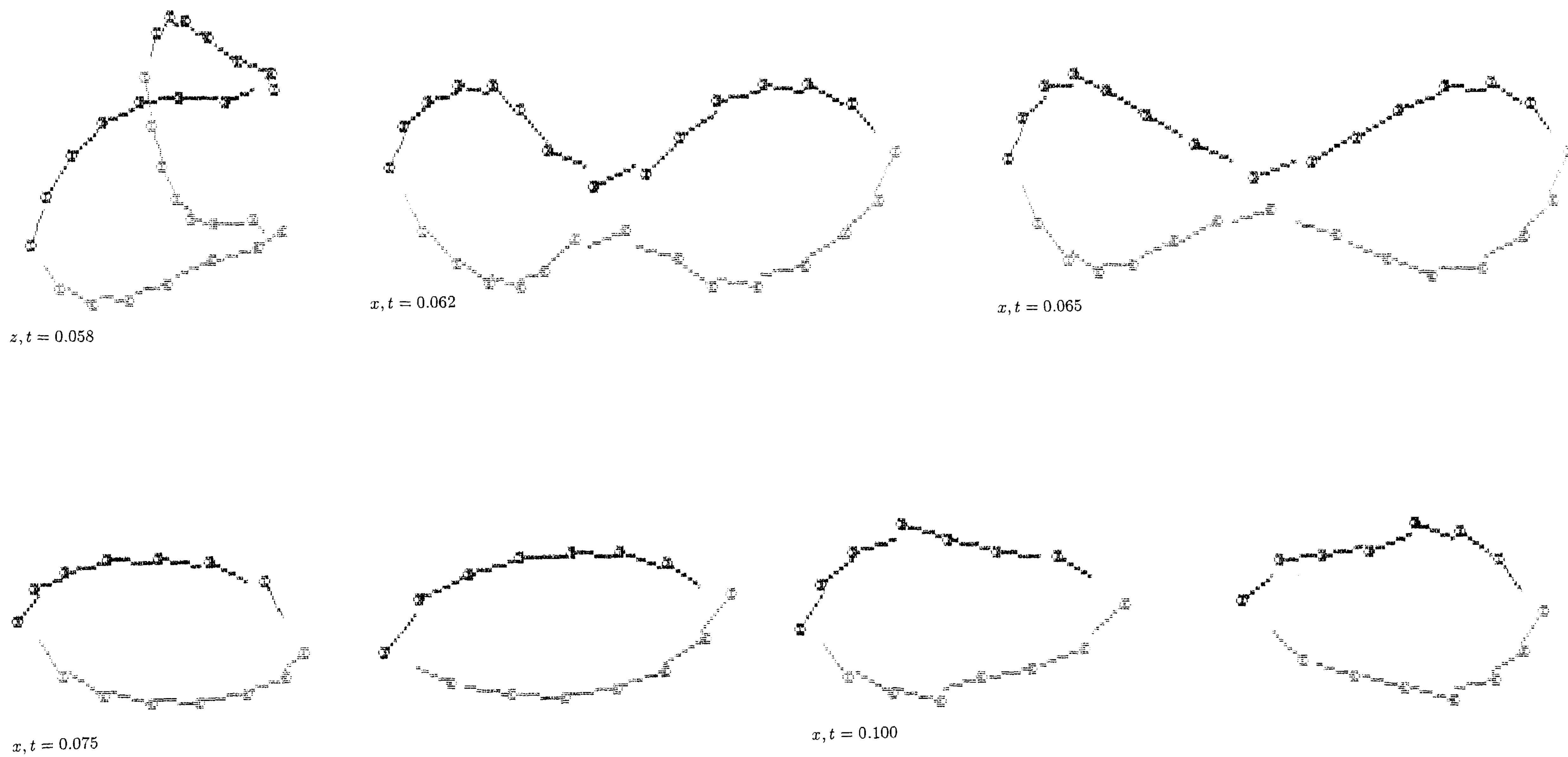



Figure 10.32 (c) (i)

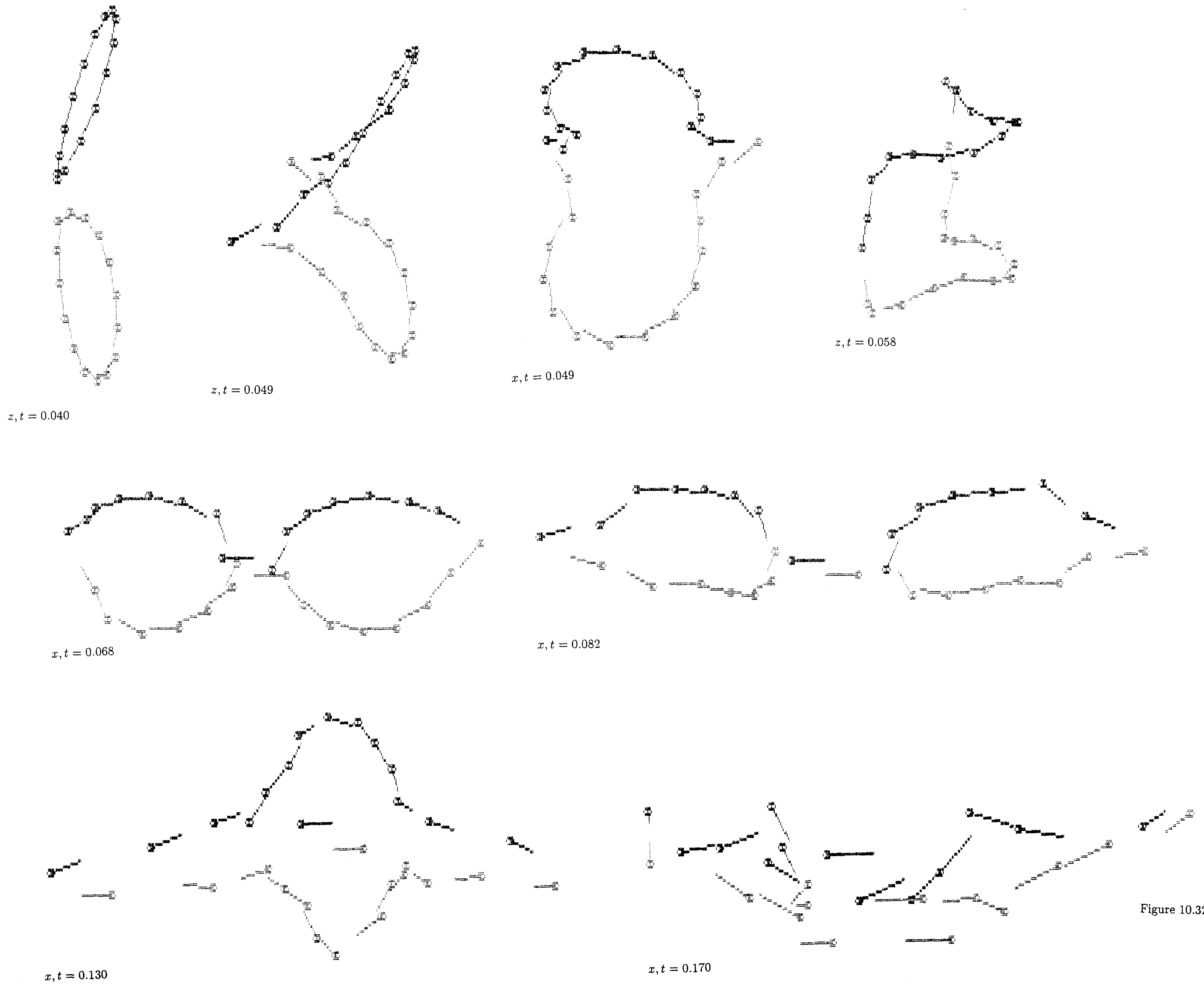



Figure 10.32 (c) (ii)

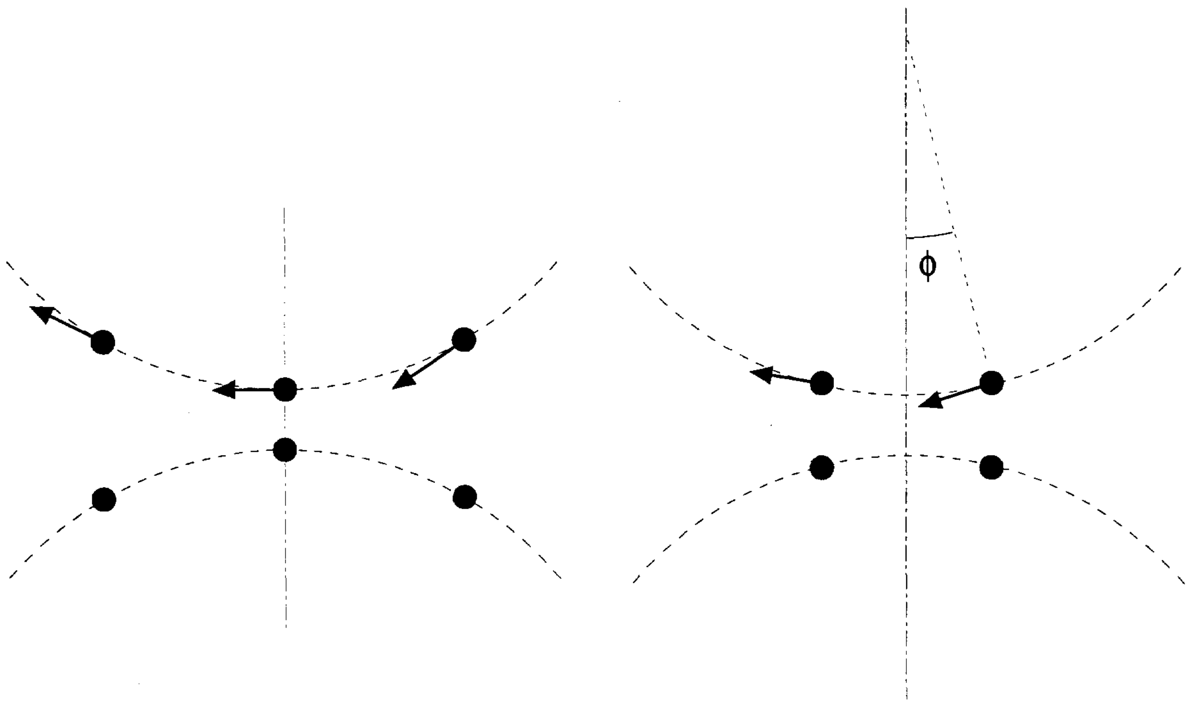



Figure 10.31: Two possbilities for the initial positions of the vortons in the configuration of fig.10.25 (seen from the left; for the lower ring only vorton locations are drawn): (a) $\phi/(2\pi/N) = 0.0$, (b) $\phi/(2\pi/N) = 0.5$.

Figure 10.32: (*see inserted sheets*) Oblique interaction of two vorton rings ($R = 1.0$, $\Gamma = 820$, $N = 15$). Vorton deformation according to (a) N-equation; (b) K-equation; (c) N+K-equation. Configuration as in (i) fig.10.31(a); (ii) fig.10.31(b). Dots indicate vorton locations, arrows indicate vorton strength vectors; the two rings are colored by different grades of black. The pictures are shown from two different points of view (compare fig.10.25): $x$ = along the $x$-axis, $z$ = along the $z$-axis. $t$ = time.

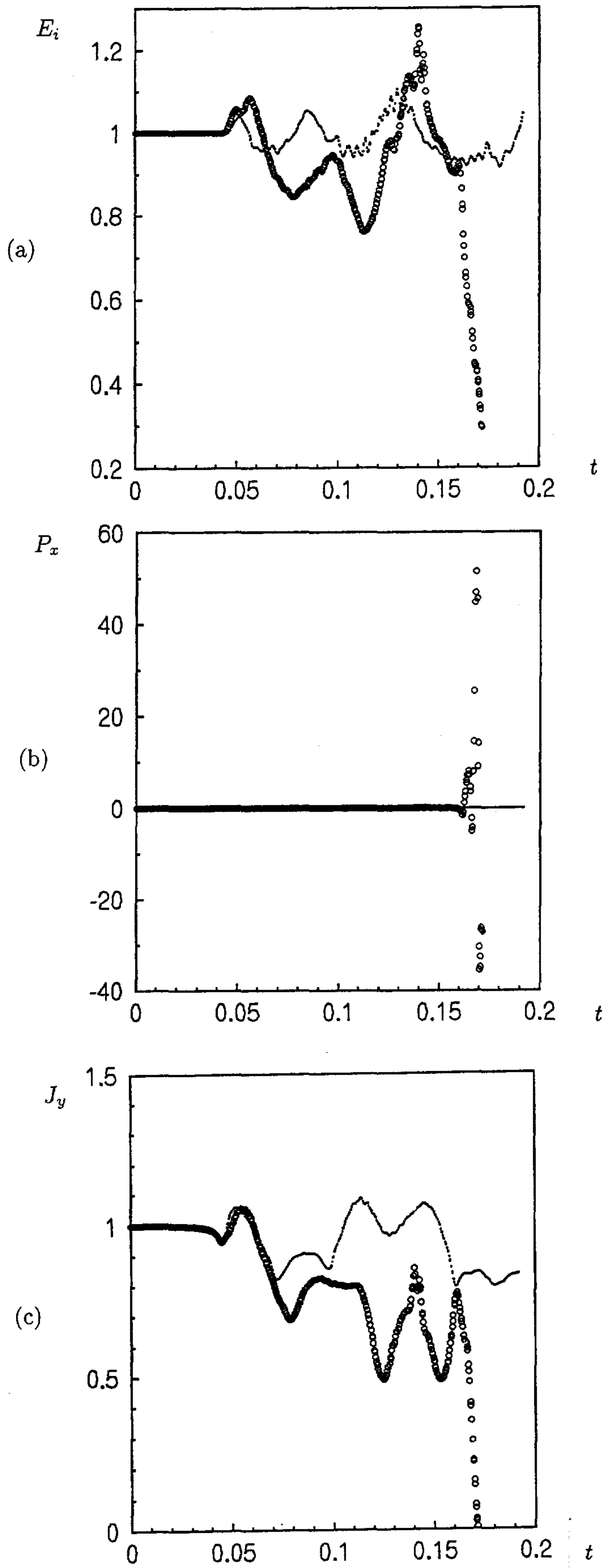



Figure 10.33: Comparison of the simulations shown in (∘) fig.10.32(a)(ii) (N-equation) and (·) fig.10.32(c)(ii) (N+K-equation). (a) $x$-component of linear momentum, $\boldsymbol{P}$, according to (9.5) (scaled with initial value); (b) $y$-component of angular momentum, $\boldsymbol{J}$, according to (9.8); (c) interaction-energy $E_i$ according to (9.10) (scaled with initial value). $t$ is time.

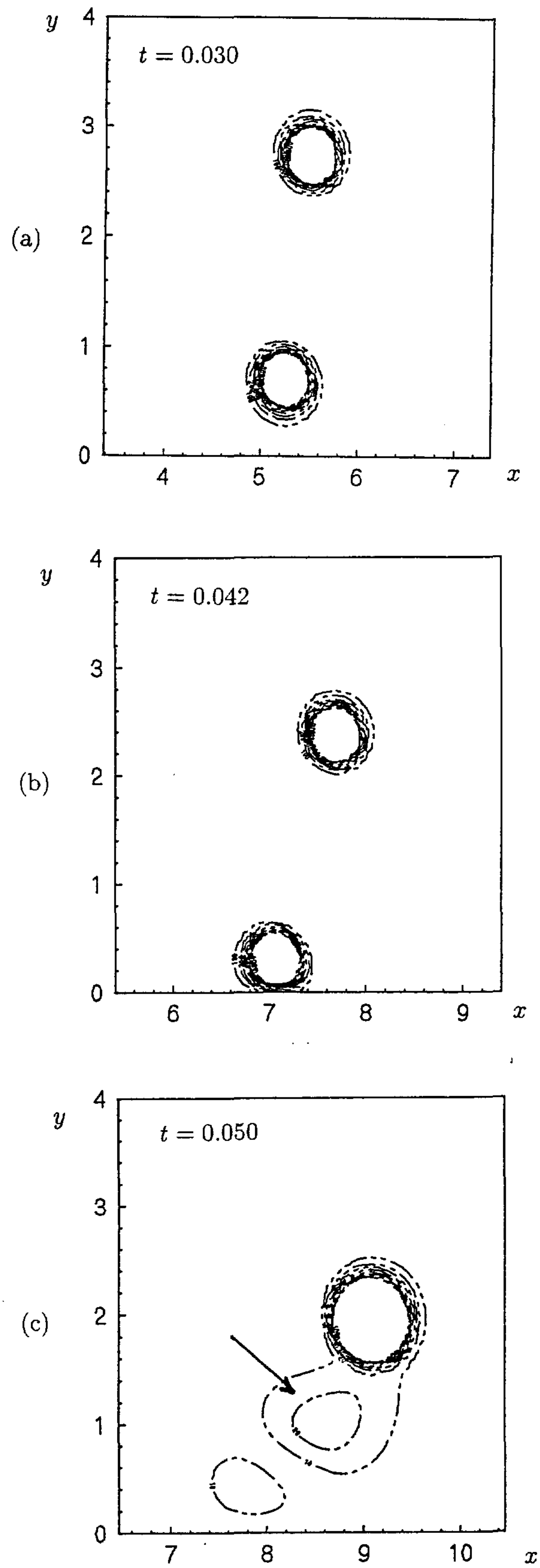
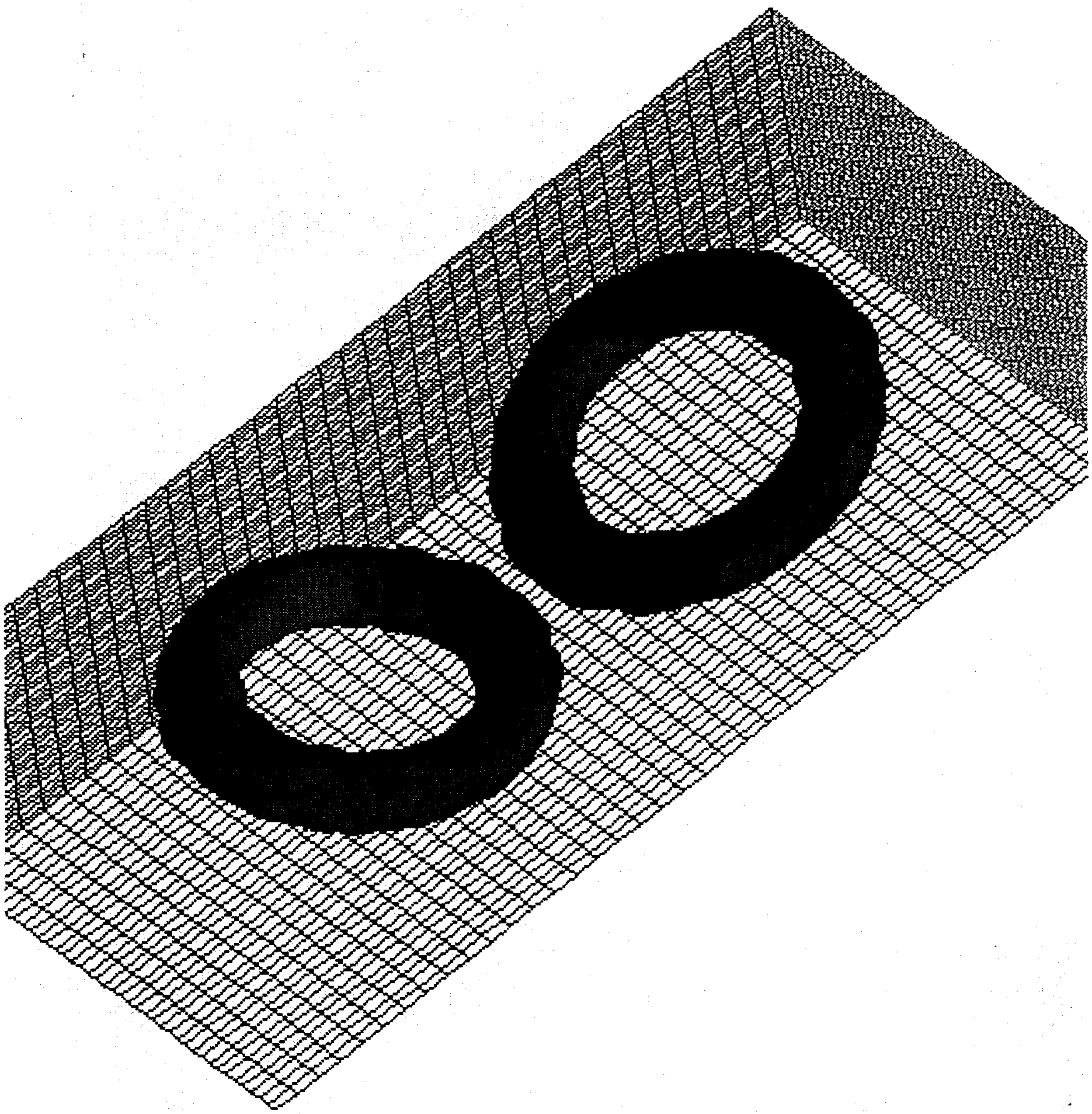
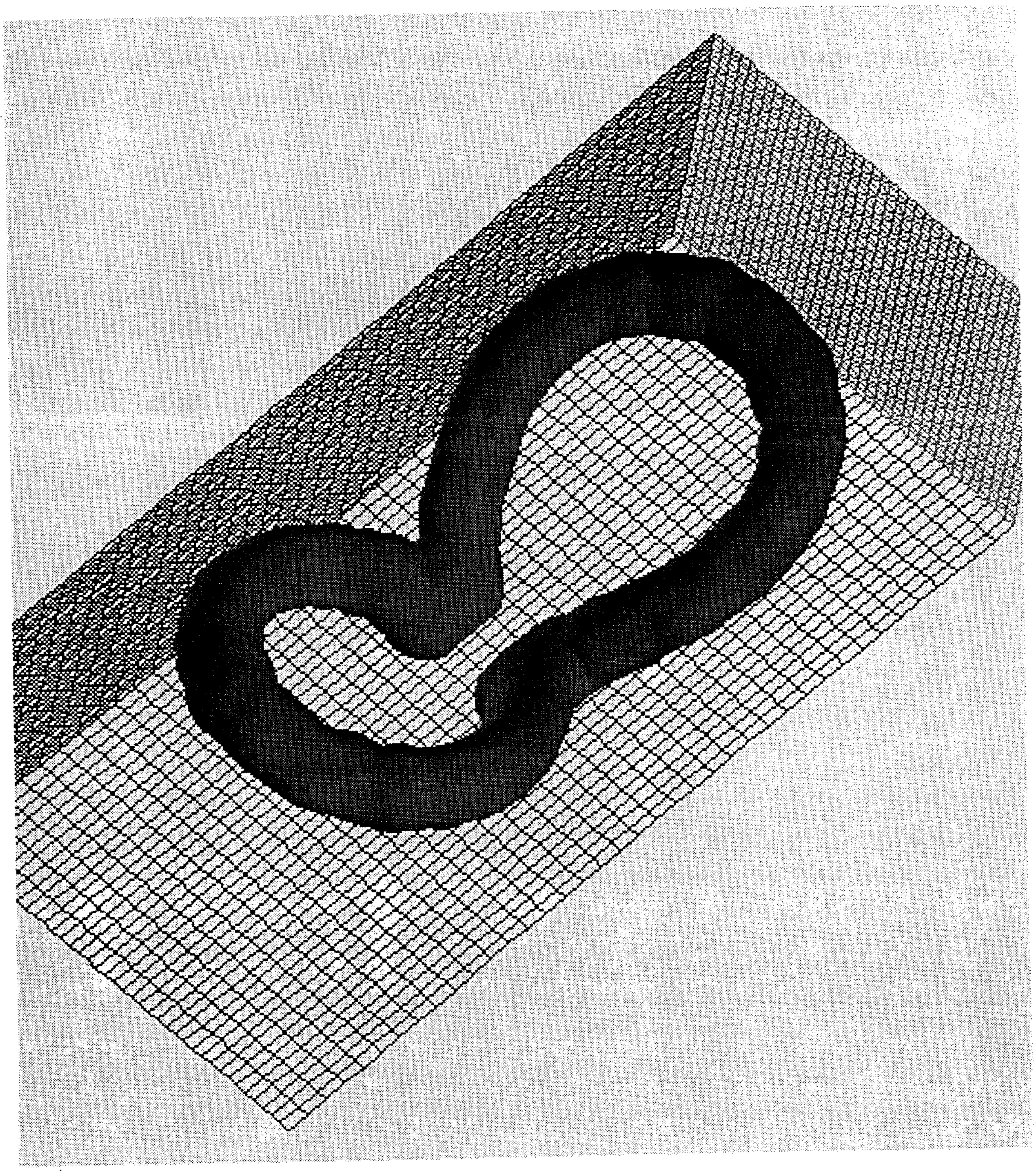



Figure 10.34: Oblique interaction of two vorton rings for the configuration of fig.10.32(c)(ii): contour plots of $|\bar{\mathbf{w}}|$. Cross-sections in $x-y$-plane for upper ring only. (a) $t = 0.030s$, (b) $t = 0.042s$, (c) $t = 0.050s$. The arrow indicates a possible thread.

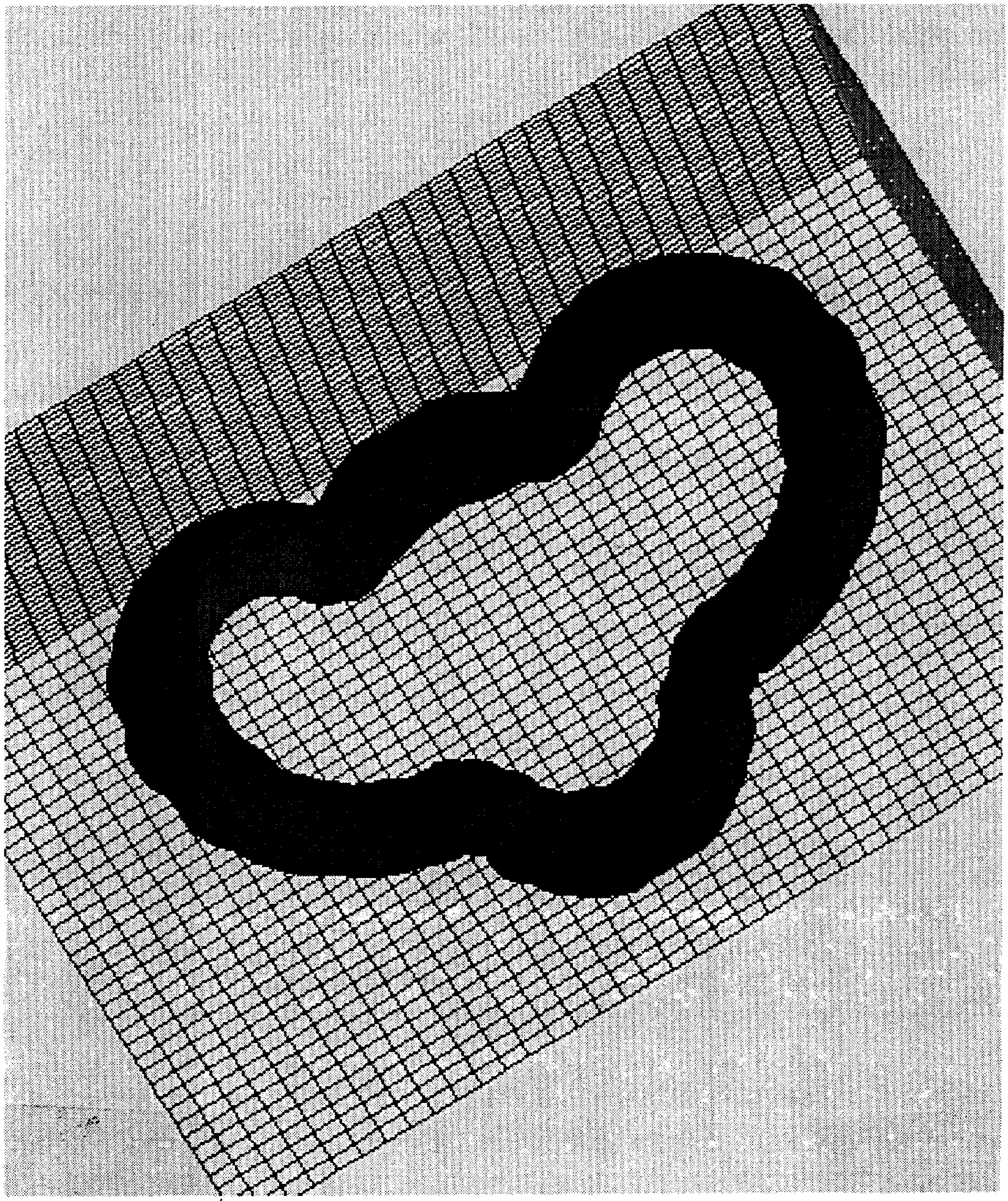


(a) $t = 0.040$.

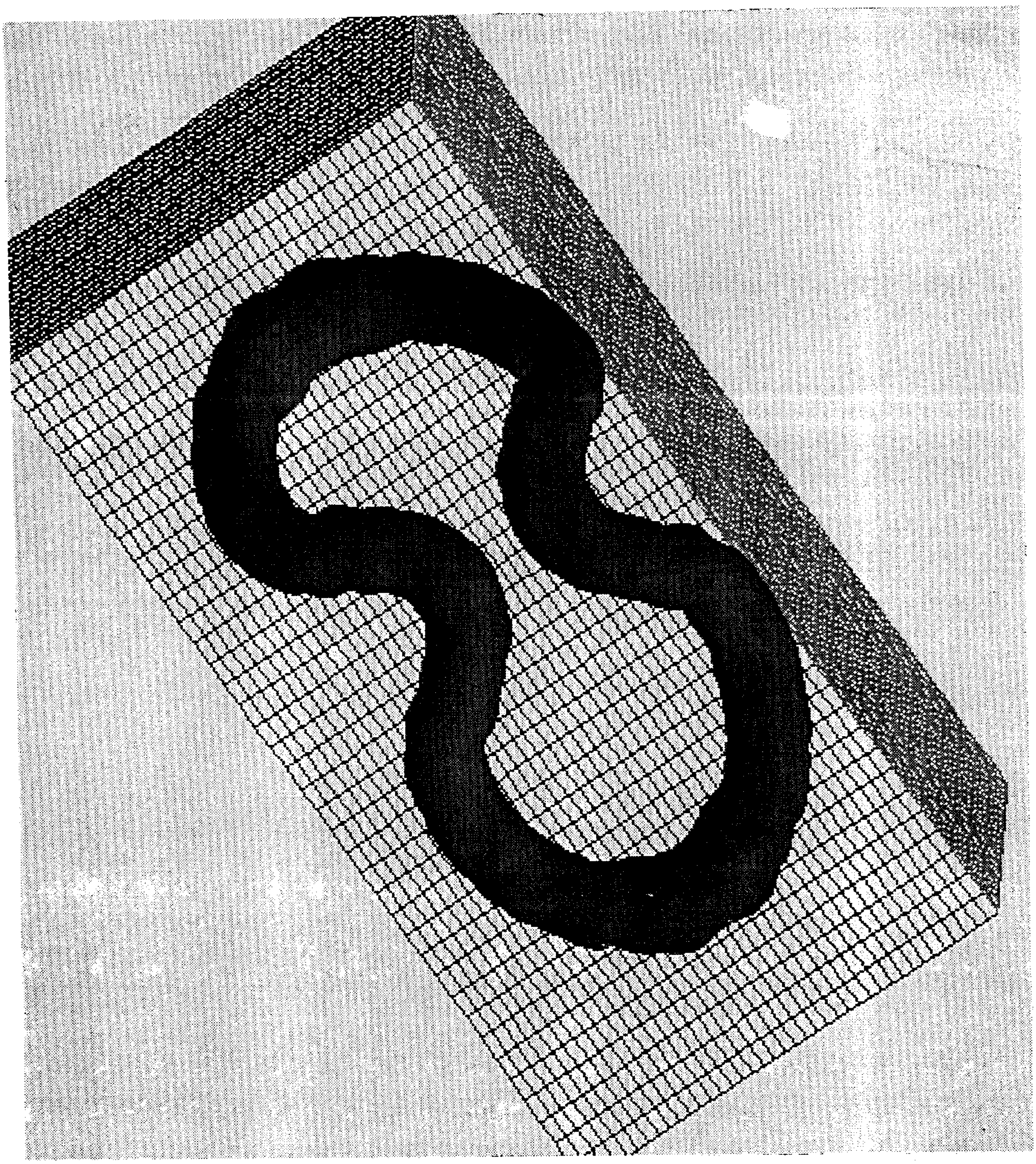
Figure 10.35: *(continued on next pages)* Oblique interaction of two vorton rings (configuration as in fig.10.32(c)(ii)): isosurfaces of $|\bar{\mathbf{w}}|$ (given by (9.18)). $t$ is time. Compare fig.10.27.



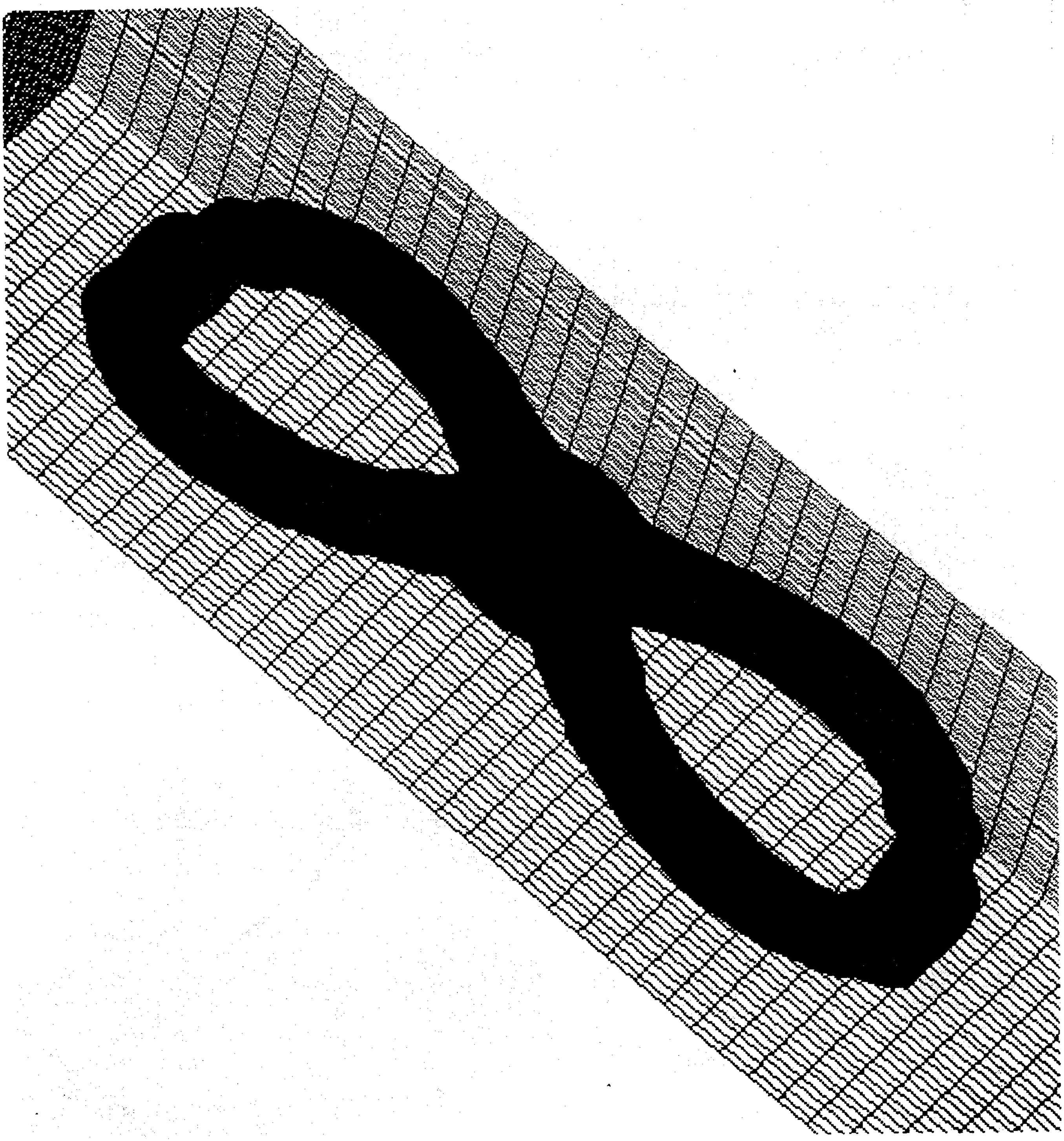
(b) $t = 0.046$.



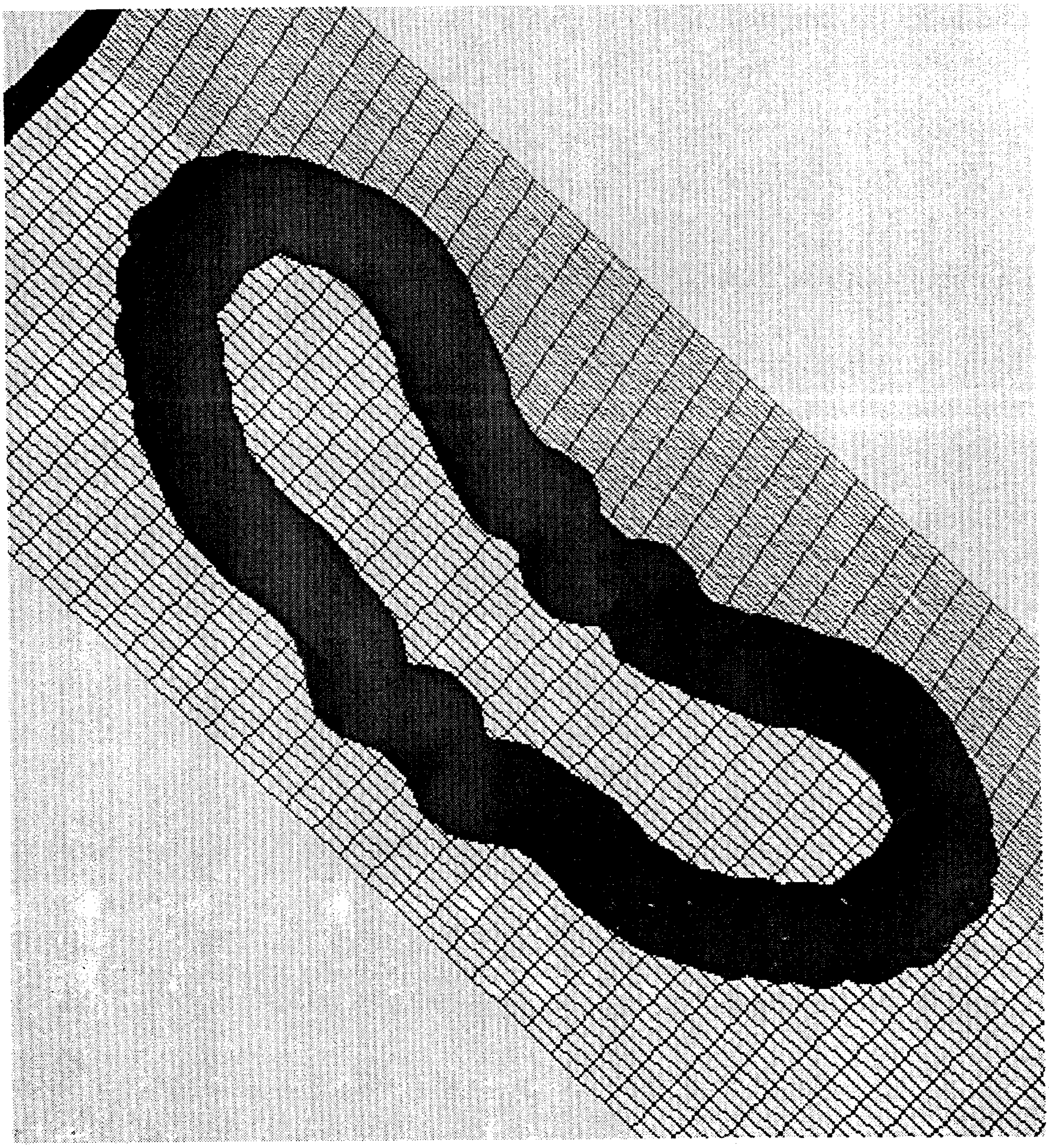
(c) $t = 0.050$.



(d) $t = 0.060$.



(e) $t = 0.070$.



(f) $t = 0.105$.

We have also investigated the susceptibility of the simulation results to the value of the angle $\phi$ as defined in fig.10.31, i.e. to the locations of the vortons. For the N+K-equation the simulations have been repeated for several values of $\phi/(2\pi N)$ between 0.0 (fig.10.31(a)) and 0.5 (fig.10.31(b)): 0.125, 0.25, and 0.375. In all cases the interaction of the two vorton rings ended in chaotic vorton motions. However, when the distance between the rings' centres $D$ was decreased to $2.5R_0$ instead of $4R_0$, we found for all three values of $\phi$ and for the N+K-equation that the behaviour appeared to be similar to that shown in fig.10.32(c)(ii) and chaotic behaviour held off. Application of the N- and K-equation to this configuration *did* lead to the appearance of chaotic behaviour, showing once again the superiority of the N+K-equation.

Finally, we discuss the consequences of the application of the "division with updating" procedure (see §9.4) to the case of fig.10.32(c)(ii). We have found that during the reconnection of the two rings the condition for vorton division was not met with and consequently no increase of the number of vortons was observed. However, division took place during the approach of the straight parts of the connected vorton rings as shown in fig.10.35(e). The division in this case appeared to have a negative influence on the development of the vorton configuration. The vortons started to show chaotic behaviour and the simulation had to be terminated.

When the initial number of vortons $N$ in each ring was increased from 15 to 16, the condition for division was satisfied. However, during the reconnection vortons were added such that a "tail" of vortons formed behind the forward-moving connected vorton rings. This tail consisted of vorton "dipoles" as shown in fig.10.32(c)(i). Due to the small distance between the vortons in these dipoles, the timestep in the simulation was seriously reduced and finally chaotic behaviour appeared.

In both cases mentioned above, the problem can be attributed to the effect of division as shown in fig.10.36. The number of vortons starts to increase quickly. Though initially the vortons remain neatly aligned, at a certain moment instability behaviour sets in. This is most probably due to the growing misalignment of the vortons caused by the crude interpolation procedure, explained in §9.4.

## 10.5 Interaction of Two Knotted Vorton Rings

### 10.5.1 Introduction

The vortex configurations treated so far show symmetry in one or more planes. As a consequence, several motion-invariants like total angular momentum and total helicity are zero and remain perfectly conserved apart from slight fluctuations around zero due to numerical errors. One of the simplest configurations in which asymmetry has essential consequences and which has non-zero helicity, is that of two knotted vortex rings as shown in fig.10.37. Besides, this configuration is one of the most elementary in which the alignment of vortex tubes can be investigated (see §C of the Interlude).

### 10.5.2 Recent Results from Literature

The configuration of two knotted rings shown in fig.10.37 already appeared in Kelvin's paper "On vortex motion" [245] of 1869 (see Kelvin's letter to Helmholtz in §3.2 and fig.4.1). However, it got little attention for many decades, presumably due to the obvious lack of experimental results. Only the advent of numerical methods renewed interest in this problem. Some recent numerical results are listed below:

- Leonard & Chua [123] studied the configuration of fig.10.37 by means of a soft-vorton method (see Appendix B), which included a "core-spreading diffusion equation" for the

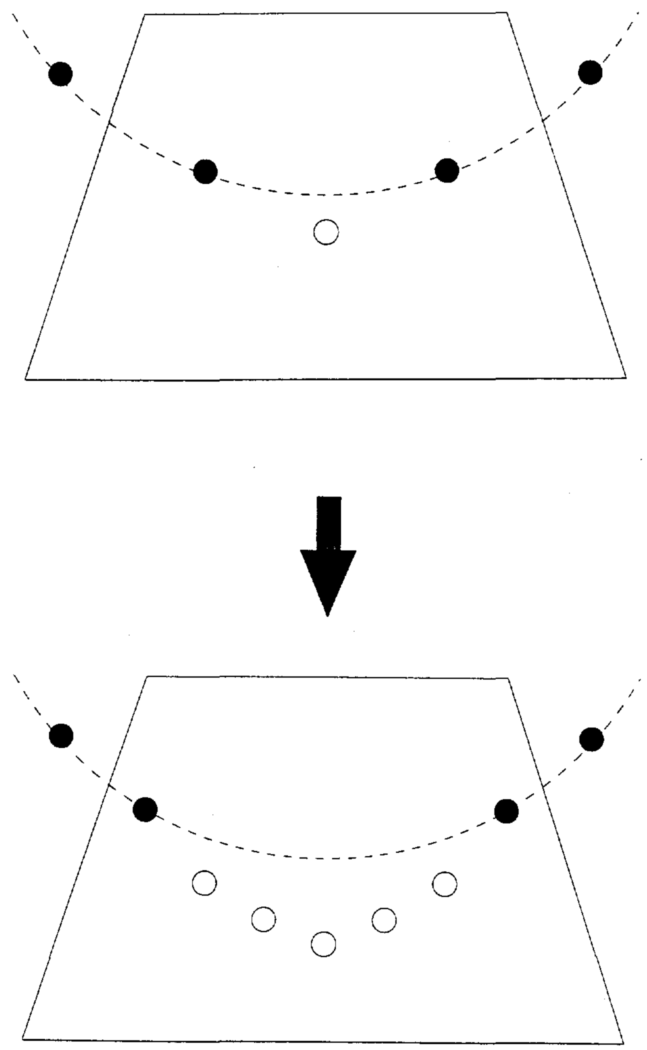


Figure 10.36: Result of vorton division (with updating) on the behaviour of two obliquely interacting vorton rings. Added vortons are indicated by open dots at the vorton locations. "Mirrored" vortons are not shown. The arrow indicates time development.

core size parameter and a viscous diffusion scheme. The cores of their rings consisted of several vortex lines comparable to the vorton ring shown in fig.10.28. Their simulation showed the formation of a "anti-parallel double tube structure". However, this structure showed no reconnection or even annihilation of vorticity.

- Aref & Zawadzki [15] applied a vortex-in-cell method (already mentioned in §10.4.2). They also observed the anti-parallel alignment of parts of the rings and claimed that these aligned parts would annihilate eahc other due to diffusion, leaving a single ring-like structure. However, this is not shown by their pictures; see fig.10.38.
- Winckelmans [283] applied a smoothed-vortex-filament method (see §7.3.1) and found the same results as Leonard & Chua. He also applied a soft-vorton method (K-equation),

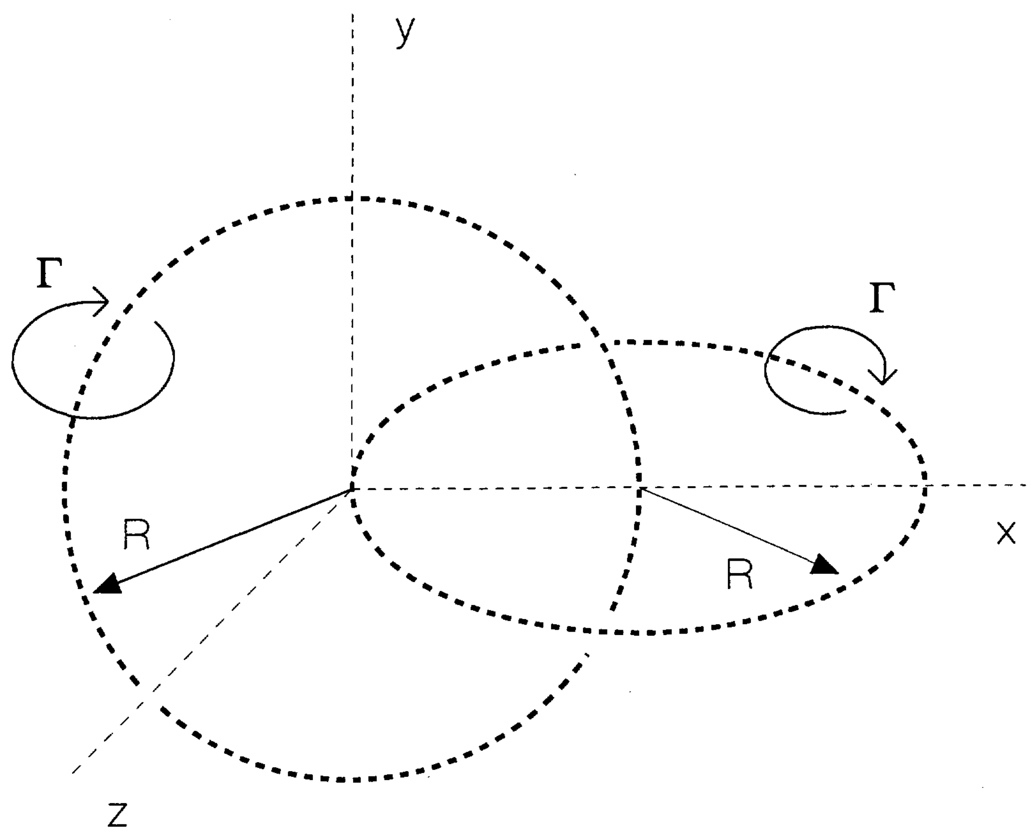



Figure 10.37: Initial configuration of two knotted vortex rings.

with and without inclusion of a vorton division scheme and his "procedure of relaxation of vorticity divergence" (see note 2 of Chapter 9). Alignment occurred but the subsequent interaction of the aligned tubes remained unclear and exhibited considerable violation of conservation of helicity. A simulation in which viscous diffusion had been included (see §10.4.2) showed no differences, due to the small convective timescale.

An even more elementary configuration to study anti-parallel alignment is the interaction of two initially orthogonally offset vortex tubes, as shown in fig.f of the Interlude. From direct numerical simulations, Zabusky *et al.* [288] suggested that the influence of the "double-layer" formed by the tubes is only local and that the topology of vortex lines contributes to limiting vortex stretching. They found that after alignment, reconnection occurred [15]. Pedrizzetti [178] simulated the interaction of two initially orthogonal vortex filaments by means of the vorton method (N-equation) and found reconnection.

[15]Zabusky and co-workers (e.g. in [288]) have also suggested that their numerical results may explain turbulence phenomena. They found a highly distorted vortex ring as debris after the process for which they suggested a similarity to the Falco ring, a structure which has been claimed to play a certain role in turbulent boundary layer flows (to be discussed in §10.6.1). The intense energy-dissipation clusters which have been found in homogeneous turbulence might be related to the regions where reconnection occurred. In these regions they observed "bursting", i.e. a sudden increase of local vorticity and dissipation.

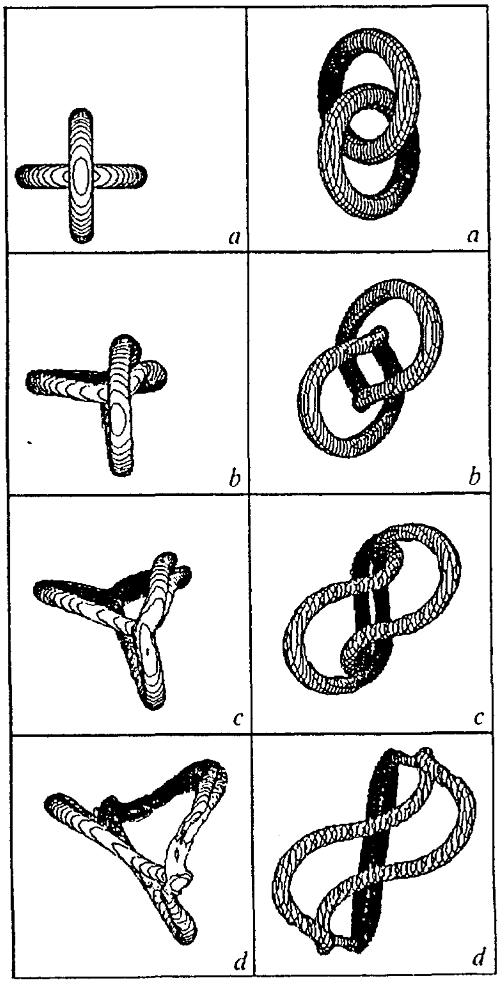


Figure 10.38: Numerical results of Aref & Zawadzki for the configuration of two knotted vortex rings as shown in fig.10.37. Isosurfaces of vorticity magnitude. Time development from top to bottom, two views are shown. From [15].

We have to conclude that the simulations mentioned above have not led to a conclusive picture of the development of two linked rings. If reconnection really takes place, we could wonder whether it is of the same kind as that of the obliquely interacting vortex rings discussed in §10.4. Kida & Takaoka [102] and Boratav *et al.* [27] have suggested that this is not the case, though an exact description of the difference appears to be still lacking.

### 10.5.3 Numerical Results

For our simulation of the configuration shown in fig.10.37, we have used two standard vorton rings ($N = 36$ for each ring) and applied all three vorton equations. For the N+K-equation we observed anti-parallel alignment as in the simulation of Aref & Zawadzki (see fig.10.38). For the other two cases, irregular behaviour started almost immediately after $t = 0$ and alignment did not occur. However, in case of the N+K-equation, the simulation also ended in a severe increase of the strength and chaotic behaviour of some vortons. No reconnection-like phenomena could

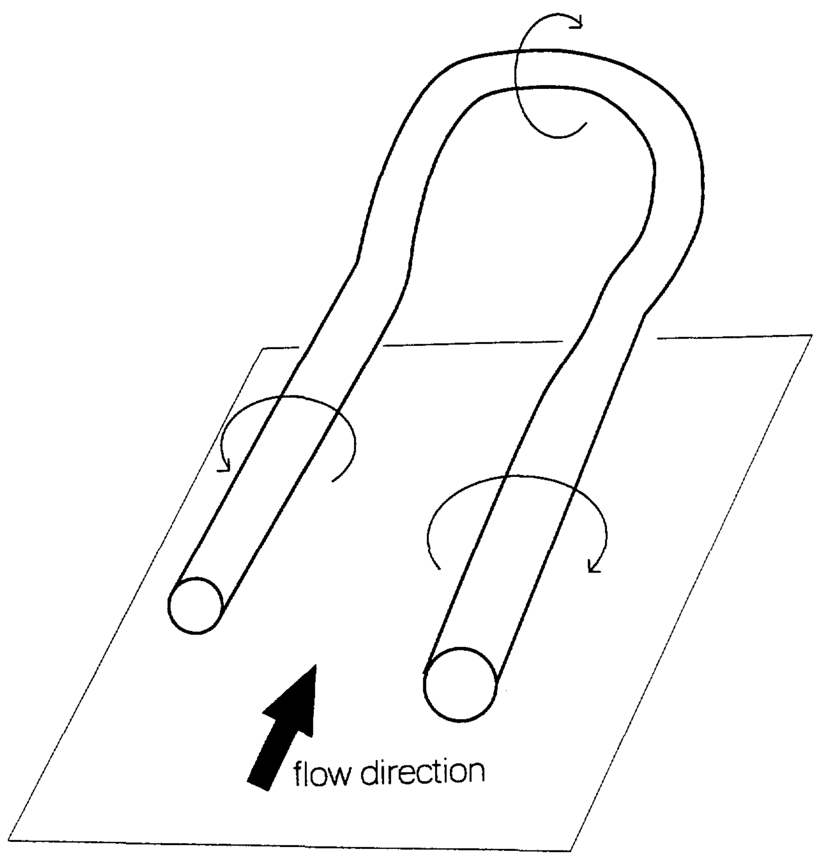



Figure 10.42: A horseshoe vortex.

First of all, we have to remark that the existence, let alone the role, of HVs is still a point of discussion. Though already proposed in the 1950s, experimental evidence of the existence of the HV only began with the flow visualizations of Head & Bandyopadhyay [74] in 1981. They showed that vertical cross-sections of elongated vortex structures made angles of 45° with the wall.

Despite ongoing controversies, the existence of HVs in TBL is nowadays generally accepted (see [68] for references; see also Smith & Lu in [107]). However, there is lack of knowledge on their formation, growth, destruction, regeneration, and contribution to gross statistics. We refer to e.g. [195] for further information. A survey of conceptual HV models can be found in Robinson's paper in [73].

Few experimens on controlled HVs have been published. Acarlar & Smith [2] studied the behaviour of HVs shed from a hemisphere in a laminar boundary layer. They concluded that between the legs of a HV low-momentum fluid is lifted up. Due to the interaction with higher-speed outer flow, secondary vortices are generated in proximity of the primary HV. These secondary vortices strongly interact with the original HV, generating chaotic structures which suddenly eject away from the wall. "These events appear very similar to the break-up stage of the burst sequence observed in turbulent boundary layers" (compare fig.10.41).

Smith *et al.* in [73] found that a viscous-inviscid interaction of a HV with the flow near the be observed.

In order to get a better view of the alignment, we show in fig.10.39 the isosurface of | $\bar{\boldsymbol{w}}$ |, given by (9.18). It shows the touching of the vortex tubes, though at this level of | $\bar{\boldsymbol{w}}$ | a hole seems to exist in the middle of the alignment area. We also observe nonuniform thickening and thinning of the tubes. Unfortunately, shortly after this time instance the simulation had to be stopped due to exponential growth of vorton strengths.

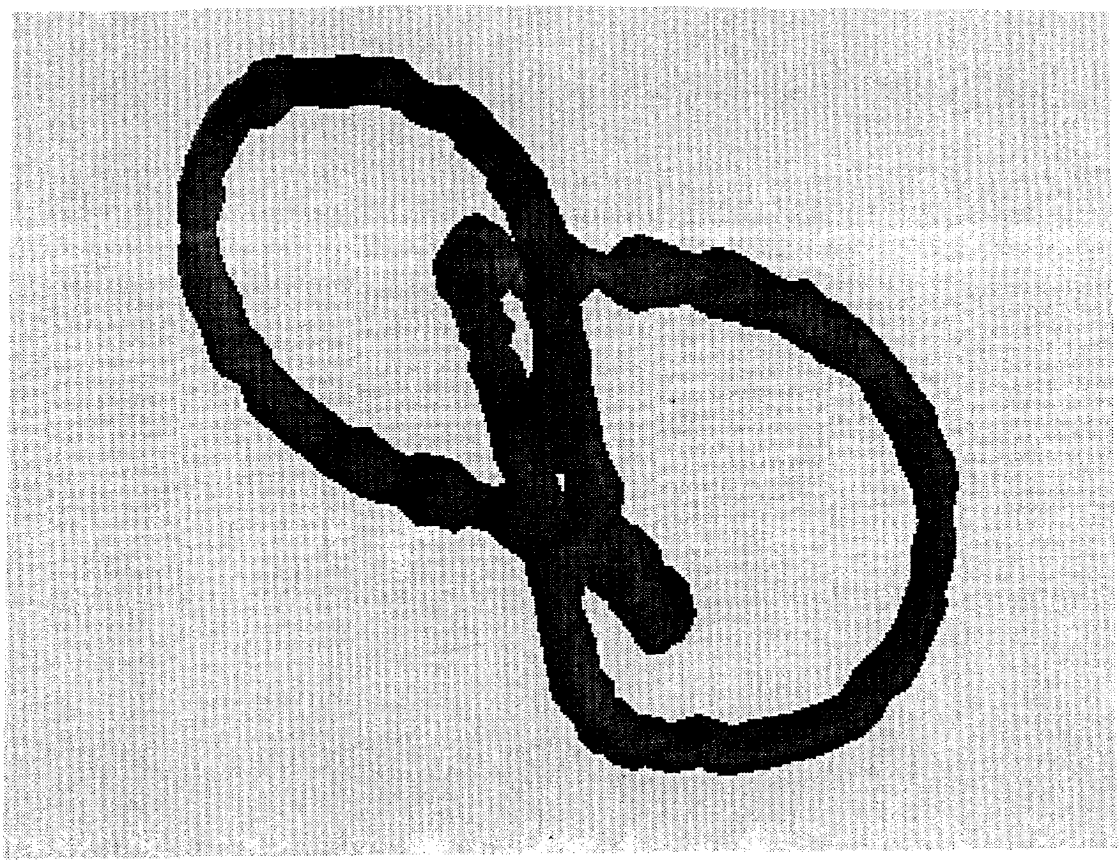


Figure 10.39: Interaction of two knotted standard vorton rings ($N = 36$): isosurface of | $\bar{\mathbf{w}}$ |. Time $t = 0.0031$ $s$.

In fig.10.40 we have plotted the development of several diagnostics for the case of the N+K-equations. We observe good conservation of all quantities up to time $t \approx 0.0032$, the moment the behaviour of the vortons starts to become chaotic. In this figure, also the influence of vorton division with updating (see §9.4) is shown. Though application of division appears to be able to extend the period of conservation of motion-invariants, it does not prevent severe violation of this conservation. This indicates that in case of division the vorton behaviour also gets irregular, though this is not immediately clear from the vorton visualizations (not presented here). Most probably the cause can be attributed to the development illustrated in fig.10.36.

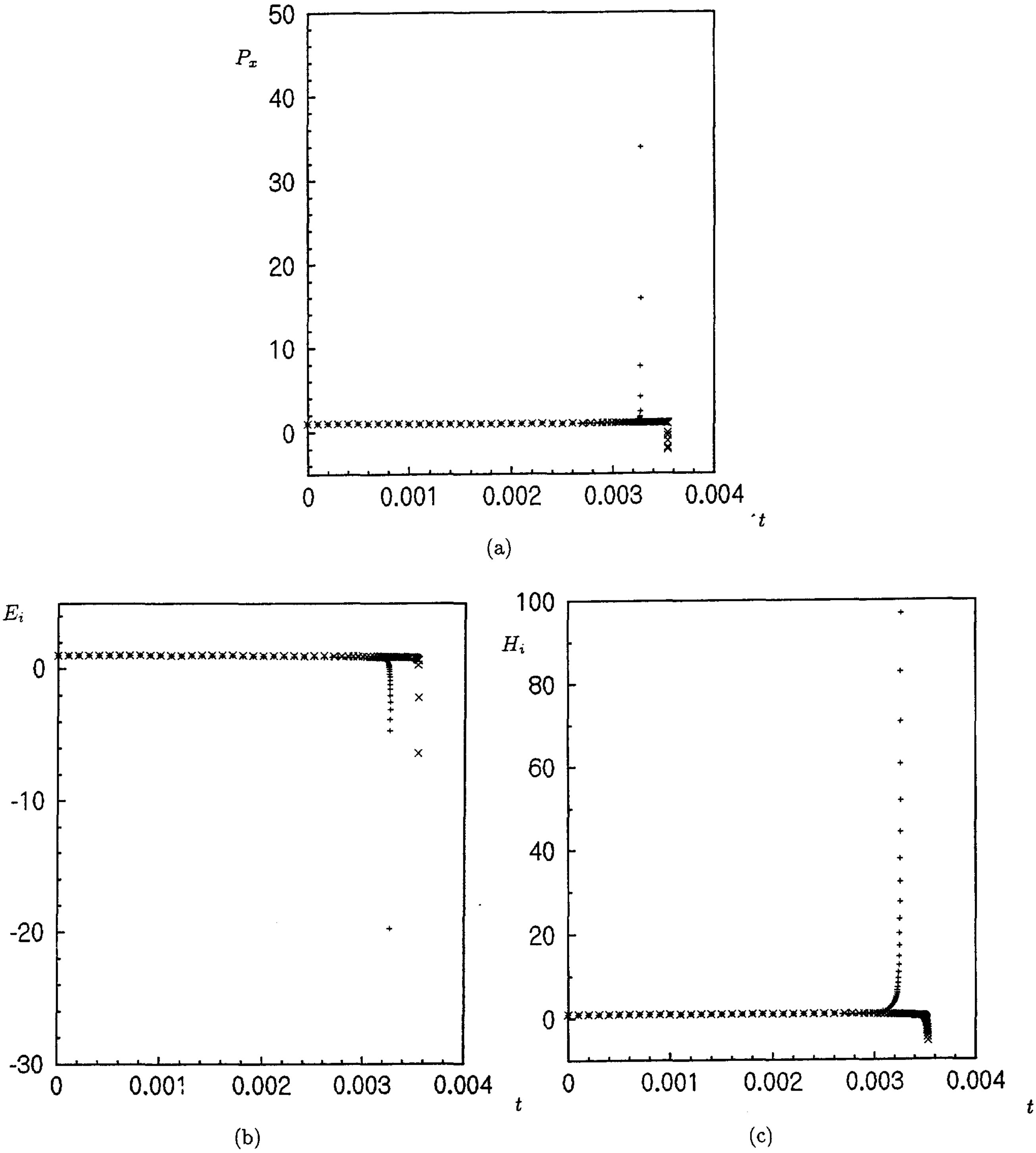



Figure 10.40: Interaction of two knotted standard vorton rings ($N = 36$): development of diagnostics. N+K-equation. (a) $x$-component of linear momentum $\boldsymbol{P}$ according to (9.5), (b) interaction-energy $E_i$ according to (9.10), (c) interaction-helicity $H_i$ according to (9.16); (all quantities are scaled with their initial values). Simulations performed (+) without and (×) with vorton division with updating. $t$ is time.

## 10.6 Single Vorton Ring in a Shear Flow above a Flat Plate

In this last section we discuss the simulation of the behaviour of a single vortex ring in the neighbourhood of a flat plate in a shear flow. Since the 1970s, mainly due to publications by Falco (see below), this configuration is regarded as a possibly useful model of a coherent structure (CS) in a turbulent boundary layer (TBL) flow. It may provide some insight into this still poorly understood turbulent phenomenon.

In §10.6.1 we will present two of the several vortical structures which have been proposed as essential elements of TBL flows, i.e. the horseshoe vortex and the vortex ring. Besides, attention is given to the possible relation between both structures. In §10.6.2 the results of our vorton simulation will be discussed. This simulation has been especially set up to investigate the influence of the so-called outer layer parameters of the shear flow on the development of this CS model.

### 10.6.1 Structures in the Turbulent Boundary Layer

In §B of the Interlude, we have seen that vortical structures, generally referred to as coherent structures, are considered to be essential elements of turbulent flows. In this thesis we will restrict attention to one of the least understood turbulent flows with regard to its "structures", i.e. the TBL flow [16].

If we limit our attention to coherent (vortical) structures in turbulent boundary layers, we already encounter a huge amount of questions. From Robinson's discussion of the objectives of turbulence-structure research [195], the following problems can be derived:

- what is the 3-D spatial character of each of the known structural features of the TBL?
- how are the various structural features related to each other in space and in time?
- what range of vortical structure topologies exists in the flow?
- what is the range of strengths (e.g. circulation) of vortex structures?
- to what extent do vortical structures play a role in determining the average production and dissipation of turbulent kinetic energy and Reynolds shear stress [17]?
- how do vortical structures form, evolve, regenerate, and die?
- what is role of the outer layer in determining details of near-wall turbulence production?
- what is the repeating sequence of events that is responsible for maintenance of turbulence, including the role of all known structures?

These questions show that research on CS has set itself a difficult task. Several models have been proposed to describe the mechanisms taking place in a TBL flow. An example is the picture which has been presented by Hinze [89]; see fig.10.41. This figure especially shows the "cyclic" process related to the phenomenon of "bursting". Bursting (see e.g. [62]) is generally used to refer to outward eruptions of near-wall fluid, resulting in a strong temporary increase of transport of momentum (or: high values of Reynolds shear stress). However, definition and usage of the term "bursting" has been confusing (see e.g. [196, Ch.12]). Numerical simulations have shown that the production of turbulence by "bursts" in the near-wall region is much more

[16] See e.g. [28], [195], and Robinson in [73] for a review of recent developments in this area.

[17] The Reynolds shear stress will be defined in §10.6.2.

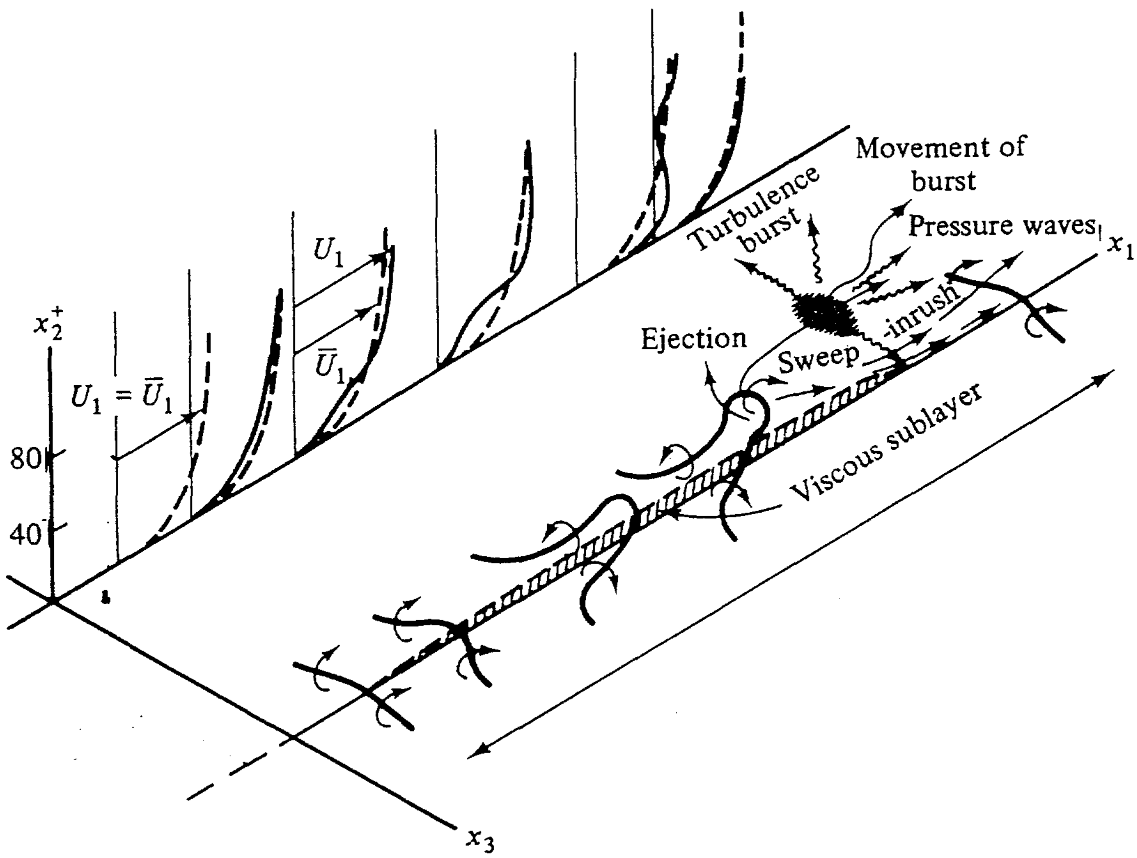



Figure 10.41: A model of the behaviour of vortical structures in a turbulent boundary layer. From [89].

intermittent in space than in time. This indicates that regions of bursting move along with the main (shear) flow.

Fig.10.41 shows how a spanwise vortex line, initially undisturbed, is deformed into a so-called horseshoe vortex. Due to an instability in the velocity profile of the flow, a burst takes place [18].

A much discussed aspect of bursting has been the parameters determining its frequency of occurrence. Does it scale with the inner or outer layer parameters? The controversy is still going on in literature (see e.g. Hussain [92] and Lumley *et al.* in [32]). We will return to this issue in §10.6.2.

Below we will treat two vortex structures which have been introduced in literature and which have been proposed as essential elements of the TBL: the horseshoe vortex, which we have already introduced, and the vortex ring, which may be related to the former.

*Horseshoe Vortices*

The horseshoe vortex (HV) has a shape as shown in fig.10.42 [19].

[18]We refer to Hinze's description in [89] for fuller details.

[19]The term hairpin vortex has been introduced for a HV of larger slenderness; here, we will regard both as having essentially the same structure.

wall caused an eruption of surface fluid which resulted in secondary vortices, probably similar to those observed by Acarlar & Smith. This suggests that the no-slip condition at the wall is of essential importance in this process.

Rogers & Moin [198] found evidence of the existence of HVs in homogeneous turbulent flows. Their results suggested that these structures do not necessarily require a wall for their formation, and that they may also develop in the presence of only a mean shear flow. However, they remarked, the shear must not be too large in order to allow the formation of HV. "The similarity in vortex structure between the homogeneous shear flow and inhomogeneous channel flow gives strong justification for the study of homogeneous 'building-block' flows as a stepping stone to understanding more complex flows".

*Vortex Rings*

The first to attribute importance to vortex rings in the TBL has been Falco (see [39] for references). He found that the outline of the TBL has the shape of large-scale bulges. At the upstream side of the bulges he visually identified coherent vortices, to which he coupled the name of "typical eddy". These Eddies, he concluded, contribute most to the production of Reynolds shear stress in the outer region and their evolution can explain the existence of streamwise vortices and horseshoe-like vortices in the TBL.

The typical eddies have been identified by Falco as a kind of vortex rings. To study their influence in boundary layer flows, Chu & Falco [39] did experiments on the interaction of vortex rings moving towards and away from a so-called Stokes' layer generated by a moving wall. This interaction led to many structural features of TBL like low-speed streaks, pockets, and HVs. The authors concluded that their results show the essential importance of vortex rings in the TBL.

With regard to the generation of typical eddies, Falco has proposed a "pinch-off" mechanism of vortex rings from Ω-shaped HV-like vortex structures; see fig.10.43. This formation process

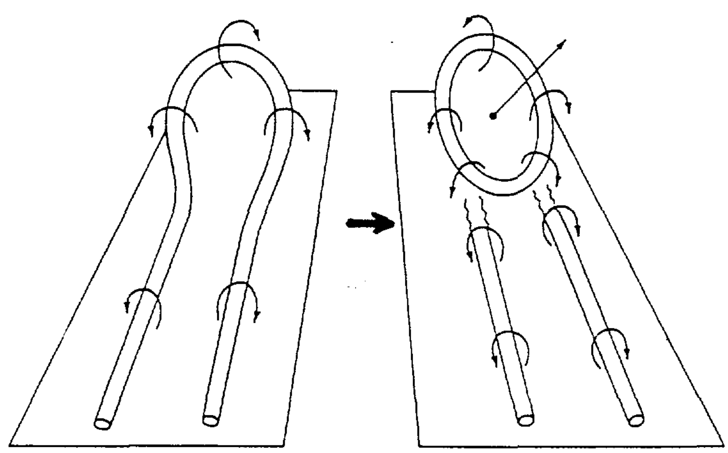


Figure 10.43: Pinch-off of a vortex ring from a Ω-shaped horseshoe vortex. Big arrow indicates time development. From [107].

was investigated by Moin *et al.* [163] who performed a numerical simulation of parabolically-shaped vortex filaments in a shear flow, modelling a HV. They applied a vortex-filament method, and hence didn't take into account vortex deformation. They indeed found pinch-off of vortex rings for parabolic shear flow profiles while linear shear profiles appeared to impede this phenomenon [20].

Evidence for the existence of vortex rings in the TBL is still lacking. For instance, Robinson [196] did not observe vortex rings or anything likewise during his diagnosis of numerical data provided by a DNS-study of a TBL. Elementary experimental studies on vortex rings near walls are still scarce. Experiments of rings in shear flows seem to be limited to that by Chu & Falco mentioned above. Walker *et al.* [274] studied the impact of a vortex ring in quiescent flow on a no-slip wall. He found the generation of secondary vorticity at the wall [21], which was ejected after some time to form a secondary vortex ring. This caused wavelike instabilities on the primary vortex ring and the flow field degenerated into smaller and smaller 3-D motions. In the end, the authors found that "the end result is an apparently chaotic flow which appears to be turbulent" and remarked a similarity with bursting in the TBL. However, they also suggested that vortex configurations in the TBL are more complicated than simple vortex rings and that the formation of secondary vorticity may be essential to understand TBL flows.

As remarked above, Falco has suggested that typical eddies may be related to horseshoe-like vortices in the TBL. However, it seems that the issue of a possible relation between typical eddies and HVs has gained only little attention from others. According to Adrian in [107] a HV can be decomposed into a vortex ring plus a mean shear plus two streamwise vortices as illustrated in fig.10.43 . The vortex rings and/or HV dominate in the outer layer and streamwise vortices dominate the wall layer. The same picture has been sketched by Klebanoff *et al.* [106], who did very extensive experiments on the transition to turbulence in a boundary layer, induced by roughness elements. They regarded the TBL as consisting of two regions: in the inner region the turbulence is generated by a complex interaction of HVs and other vortical structures induced by the obstacles; in the outer region the HV generate turbulent vortex rings. The latter may be responsable for the bulges the authors observed at the edge of the boundary. However, they found no convincing evidence for the existence of $\Omega$-shaped HVs as suggested by Falco, but they didn't exclude them either. The authors concluded that the eddies in the outer region do not (directly) contribute to the transition to turbulence in the TBL, whereas the HVs *are* intrinsic to this process and to developing turbulence.

### 10.6.2 Vorton Simulations

Though we have to conclude that the existence and the role of vortex rings in the TBL is still uncertain, we try to make a contribution to the understanding of this role by considering the elementary configuration sketched in fig.10.44: a single standard vorton ring in a shear flow above an infinitely extended flat plate. We realize that we will not be able to simulate phenomena related to the no-slip condition at the wall (e.g. generation of secondary vorticity). Therefore, it is better to investigate a situation like the one presented here in which these phenomena are supposed to be absent or of minor importance.

[20] Morrison *et al.* [164], however, concluded that the pinching-off as found in simulations by Moin *et al.* [163] can only be a relatively rare phenomenon.

[21] A similar study by Lim [127] showed the same result.

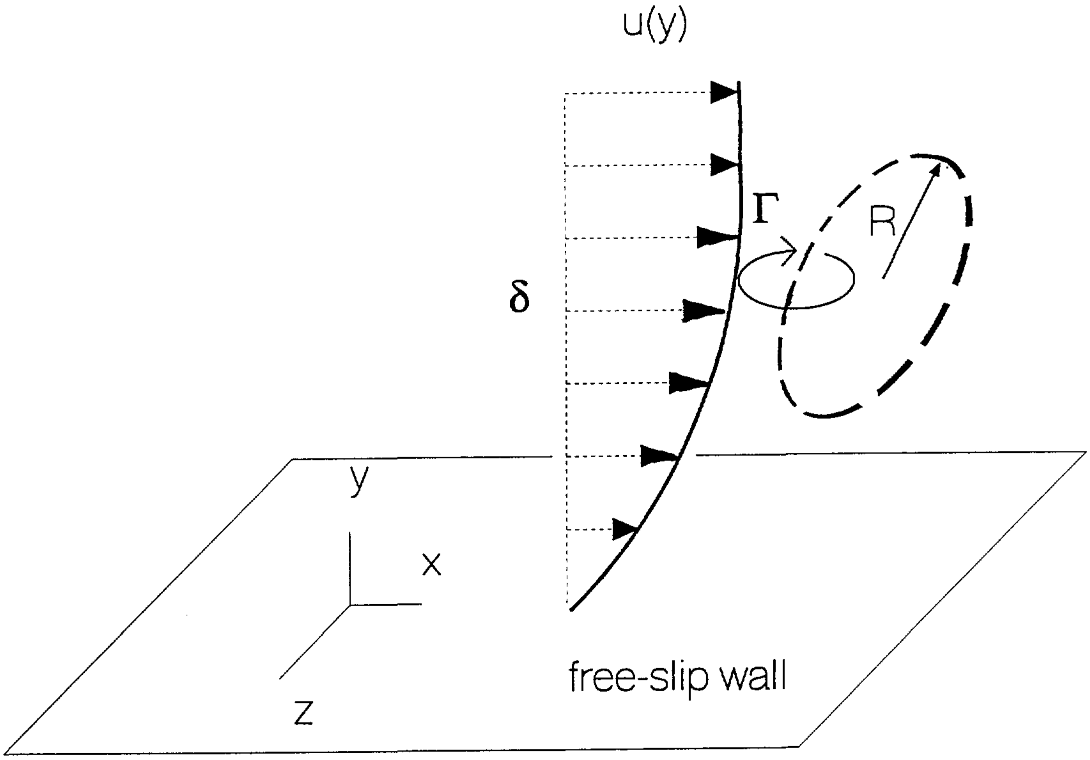



Figure 10.44: Initial configuration for the single vorton ring in a shear flow above a flat plate.

The ring is initially at an angle of 45° with the $x-z$-plane and its center is located at $y = H$. Its velocity is directed away from the plate. The flat plate is simulated by means of a mirrored vortex ring (compare fig.7.1). In our simulations we will take the $x-z$-plane as (infinitely extended) boundary plane. The parameters of the mirrored vorton $\alpha'$ of a vorton $\alpha$ are given by:

$$(r_{\alpha'})_x = (r_\alpha)_x \ , \ (r_{\alpha'})_y = -(r_\alpha)_y \ , \ (r_{\alpha'})_z = (r_\alpha)_z \tag{10.6}$$

and

$$(\gamma_{\alpha'})_x = -(\gamma_\alpha)_x \ , \ (\gamma_{\alpha'})_y = (\gamma_\alpha)_y \ , \ (\gamma_{\alpha'})_z = -(\gamma_\alpha)_z \tag{10.7}$$

where the index $x$, $y$, and $z$ indicate the components of the vectors. These relations provide a free-slip condition, since the tangential velocity at the plate will generally not be zero.

The shear flow is represented by a one-dimensional velocity profile $\boldsymbol{v}_s = (u(y), 0, 0)$. The results by Moin *et al.* [163] (see §10.6.1) suggest that a linear shear flow is not consistent with the formation of vortex rings in shear flows. Therefore, we have chosen the following profile [22]:

$$u(y) = U \tanh \frac{3y}{\delta} \tag{10.8}$$

In this profile the quantity $\delta$ can be interpreted as the height of the boundary layer and $U$ as the outer layer velocity since, due to the factor 3, the velocity $u(\delta) = 0.995U$. For $\delta$ we take the value 1.

[22]The profile does not need to have an inflection point, as has been shown by Kim in [53]. The profile used here has already been proposed by Novikov [170].

The simplest manner to include a shear flow into the vorton equations is to take into account only the advection of the vortons by the local shear velocity $\boldsymbol{v}_s$. However, we have to realize that the shear flow will also cause vortex deformation of the vortex ring and that the shear flow profile in its turn will be changed due to the velocity field induced by the vortex ring.

For a consideration of the importance of inclusion of these two phenomena in a simulation, we refer to an estimation by Aref & Flinchem [12] of the effect of the shear flow on a vortex filament. Their theoretical consideration led to the conclusion that only taking into account advection is a valid approximation to order $a/\Delta$ where $a$ is a measure for the core of the filament and $\Delta$ is a measure of the shear profile height (comparable to our parameter $\delta$ in (10.8)). However, they remark that the estimation becomes invalid when transverse oscillations of the filament introduce other length scales. This means that the approximation may become invalid in time.

To fulfil the condition given by Aref & Flinchem mentioned above, we have to require:

$$\frac{R_c}{\delta} \ll 1 \tag{10.9}$$

where $R_c$ the core size of the vortex ring. As shown in §10.1.1, for a vorton ring $R_c$ may be taken proportional to the distance between the vortons in the ring. Consequently, the number of vortons has to be as large as possible in order to be sure that taking into account only advection is sufficient for reliable simulations. Since we use a relatively small numbers of vortons in our simulation, we have added the effect of vortex deformation due to the shear flow (as has been proposed by Novikov [169]). This means that, in the N+K-equation, $(\dot{\boldsymbol{\gamma}}_\alpha)_x$ has been extended with $(\dot{\boldsymbol{\gamma}}_\alpha)_y \partial u/\partial y|_{y=(\boldsymbol{r}_\alpha)_y}$ and $(\dot{\boldsymbol{\gamma}}_\alpha)_y$ has been extended with $(\dot{\boldsymbol{\gamma}}_\alpha)_x \partial u/\partial y|_{y=(\boldsymbol{r}_\alpha)_y}$. The change of the shear flow profile due to the vortex ring will not be included, i.e. the function $u(y)$ remains unaltered.

Fig.10.45 shows the development of the configuration of fig.10.44 for a standard vorton ring in case of the N+K-equation. We observe four stages:

1. the ring moves away from the plate, at the same time deforming into a non-circular (somewhat elliptical) shape and rotating along its horizontal axis;

2. due to its rotation the ring moves into the direction of the plate;

3. having approached the plate closely, part of the deformed ring is pinched-off as a smaller ring-like vortex structure, which starts to move away from the plate;

4. the part which has remained near the plate starts behaving chaotically and the simulation has to be stopped.

Our purpose is to investigate the possible existence of burst-like phenomena in this simulation and the possible influence of the outer parameter $U$. To this end, we consider as a

Figure 10.45: (*see inserted sheets*) Development of the configuration of fig.10.44 for a standard vorton ring ($N = 18$). Two views of the same simulation are given: (a) view along the $x$-axis, (b) view along the $z$-axis. $t$ is time.

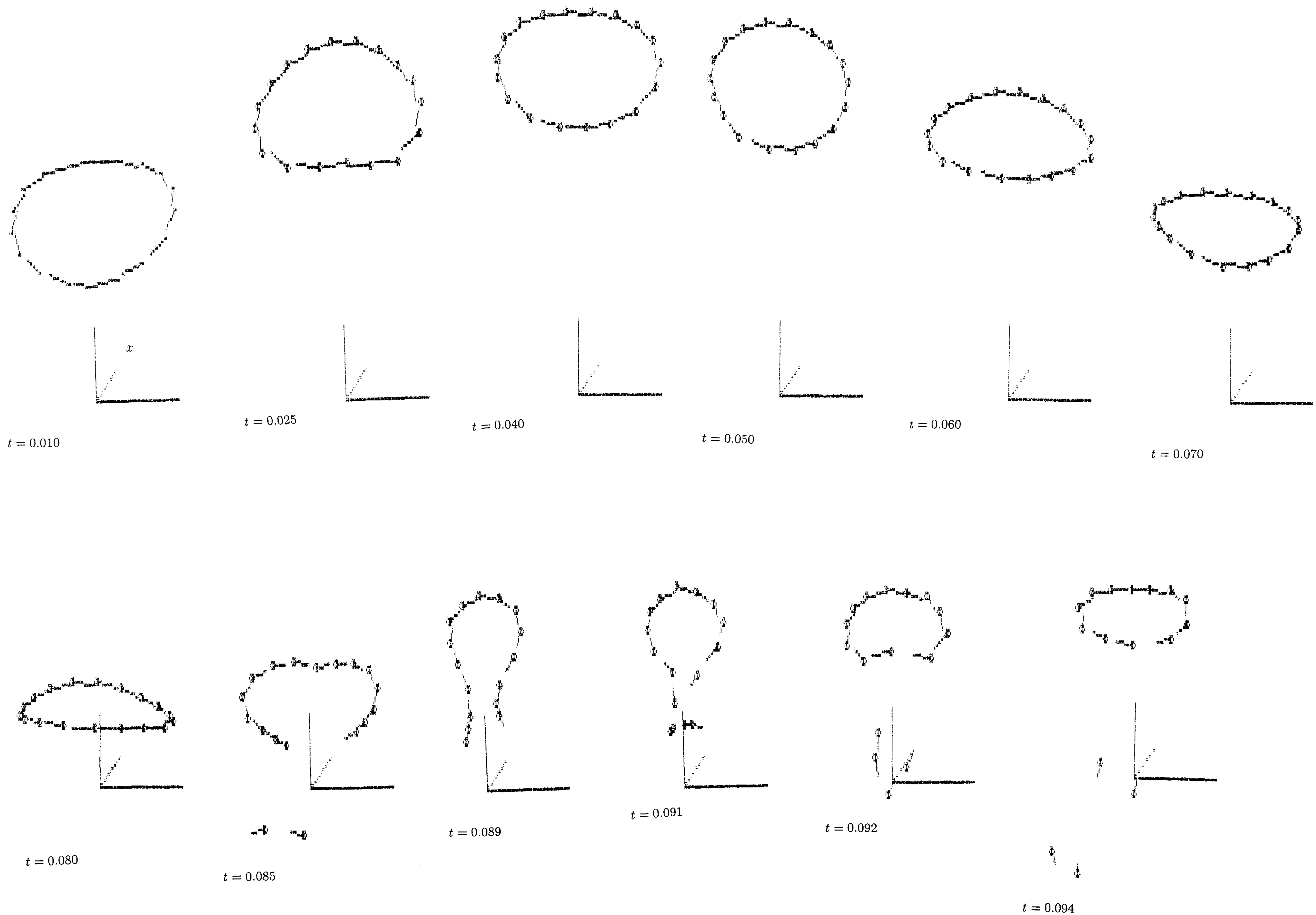



Figure 10.45 (a)

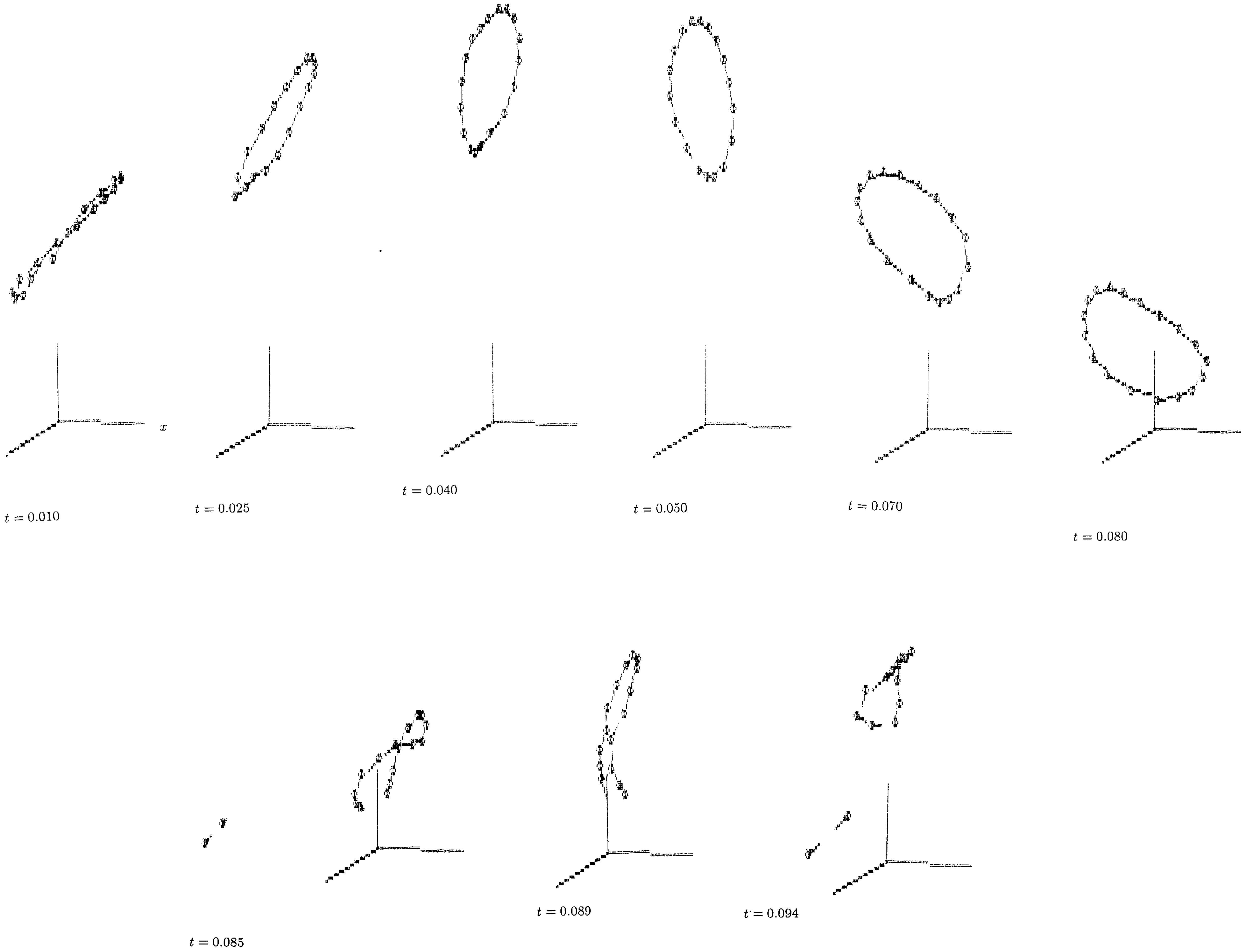



Figure 10.45 (b)

diagnostic the quantity given by $uv$, where $u$ and $v$ are the $x$- and $y$-component of the velocity field $\boldsymbol{v}$, respectively. This velocity field consists of the vorton field given by (8.10) and the shear flow field $\boldsymbol{v}_s$ mentioned above. The quantity $uv$ will be called the Reynolds shear stress [23] and it indicates transport of momentum. Since bursts are supposed to transport momentum in the positive $y$-direction, we only regard the components given by $u > 0, v > 0$ and $u < 0, v > 0$. They have been calculated at the points of a grid of height $\delta$ and extending sufficiently far into the $x$- and $z$-direction. Of all grid points, the maximum value of the Reynolds shear stress is calculated [24]. In fig.10.46 these values are plotted against time for the case $U = 100$.

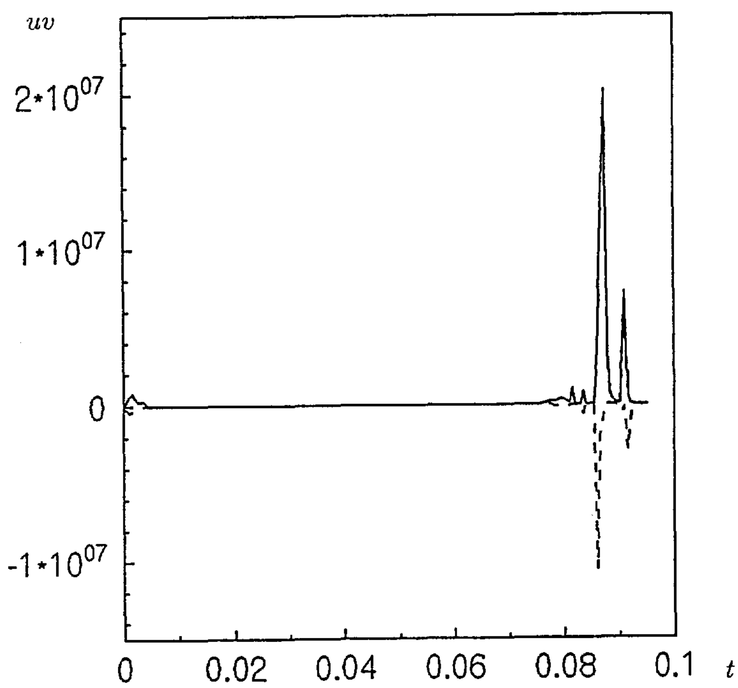



Figure 10.46: Configuration of fig.10.44 for a standard vorton ring ($N = 18$): maximum values of Reynolds shear stress $uv$ ($v > 0$, (—) $u > 0$ or (- - -) $u < 0$) vs. time $t$. $U = 100$ (see (10.8)).

We observe an intense increase of both components of the Reynolds shear stress occurs at about the same moment, which will be indicated by $t'$. In table 10.1 we have plotted the dependence of time $t'$ up to bursting on the characteristic velocity $U$ of the shear flow (see (10.8)). We conclude that the outer flow parameter $U$ constitutes a characteristic of this model of the TBL. However, it may be clear that it would be rather premature to conclude that the burst frequency in a TBL is determined by the outer flow.

[23] In turbulence, the Reynolds shear stress is usually defined as the correlation between velocity fluctuations, i.e. $\overline{u'v'}$.

[24] The contributions to the Reynolds shear stress from the vortons near the wall which start to behave irregularly at the end of the simulation shown in fig.10.45 have not been included.

| U | t' |
|---|---|
| 90 | 0.094 |
| 95 | 0.090 |
| 100 | 0.086 |
| 105 | 0.081 |
| 110 | 0.071 |
| 115 | 0.069 |

Table 10.1: Configuration of fig.10.44 for a standard vorton ring ($N = 18$): shear flow velocity $U$ vs. time to burst $t'$ (see fig.10.46) .

Chapter 11

# Discussion of Vorton Results

In this chapter, the results obtained from the numerical simulations presented in Chapter 10 will be discussed against the background of the requirements we have posed in §7.2 on vortex methods. In §11.2 we separately discuss the simulation of §10.6 (a single vorton ring in a shear flow over a flat plate) and its relevance for research on coherent structures (CS). Finally, in §11.3, conclusions are summarized regarding the applicability of the vorton method which has been used in this thesis. Besides, suggestions are made for possible improvements of the method.

While reading the discussion presented below, the reader has to bear in mind that we have concentrated on two main questions:

- Is it possible to obtain an adequate [1] representation of vortex structures by means of vortons (*in casu* vorton rings representing vortex rings)?
- Can the deformation and interaction of vorton rings be regarded as an adequate representation of physical vortex ring phenomena [2]?

For convenience, we will refer to our vorton simulations by means of the numbers of the sections in which they have been treated:

- §10.1 = single vorton ring
- §10.2 = single pseudo-elliptical vorton ring
- §10.3 = two head-on colliding vorton rings
- §10.4 = two obliquely interacting vorton rings
- §10.5 = two knotted vorton rings

## 11.1 Satisfaction of Vortex Method Requirements

### 11.1.1 Divergence-free Vorticity Field

As discussed in §9.2, this requirement on vortex methods is fulfilled due to our derivation of the vorton vorticity field. As a consequence we have been able to remove the inconsistency between the N- and K-equation, as described in §9.3.

We have compared simulations for which the N-equation and the K-equation have been applied with those for which the N+K-equation has been applied. From the simulations in §10.2 (fig.10.8), §10.4 (fig.10.32), and §10.5, we conclude that application of the K-equations

[1] By "adequate" we mean a representation which at least shows qualitative characteristics similar to those which have been found by experimental investigation.

[2] These phenomena have been mentioned in §9.1.

gives unreliable results [3]. From the simulations in §10.4 (fig.10.32 and fig.10.33) and the results mentioned in §10.5, we conclude that the N+K-equation is to be preferred above the N-equation, i.e. the equation originally proposed by Novikov [168].

### 11.1.2 Correct Modelling of Continuous Distributions of Vorticity

Regarding the representation of a vortex ring by means of a vorton ring, we observe that a core can be attributed to the vorton ring, in which the distribution of vorticity agrees qualitatively with the (scarce) experimental results on this issue (see fig.10.2). At first glance, this result may seem surprising, since vortons are generally regarded as 3-D point-vortices with zero core size [4].

The core prevents the vortons from approaching each other to arbitrarily close distances (see §10.3, fig.10.16). This result weakens one of the arguments against the vorton method, i.e. its failure when vortons approach to small distances.

The core size, however, depends on the distance between the vortons in a ring (see fig.10.3). This means that the number of vortons is restricted by the quantitative characteristics (i.e. its velocity, radius, circulation) of the ring that we want to simulate. In §10.4, we have seen that this may lead to a rather small number of vortons in a ring and to a clash with the requirements of numerical accuracy which we would like to impose. However, our simulations have shown that even for such small numbers ($N < 18$), agreement with experimental results remains acceptable (see fig.10.35). The fact that $N$ has to be a specific integer, however, limits the possibilities of imposing initial conditions.

The fact that the core size of a vorton ring is proportional to the ring's radius, means that stretching of a vorton ring, or more generally a vorton tube, is accompanied by an increase of the core size. Physically, this seems incorrect. This result has been recognized before by others. A vorton division procedure has been proposed to avoid this phenomenon. However, our implementation of vorton division has brought to light some serious drawbacks of this procedure (see the discussion below in §11.1.5).

The discrete representation of continuous vortex configurations, as done in the vorton method, may be criticized. From fig.10.1(b) and fig.10.2 in §10.1 we have found that the distribution of vorticity for a vorton ring in azimuthal direction (i.e. along the torus) is not homogeneous. However, no serious negative consequences seem to be related to this as long as cores do not "touch" [5].

Another important drawback of the discrete representation appears to be the possibility of vortons losing their alignment once vorton tubes approach closely. In fig.10.32 we have seen one example of this behaviour: closely approaching vortons tend to form "dipoles" and leave the main structure. In this regard, vortex-filament methods have to be preferred.

[3] We have to add that this result may depend on the time steps used. However, our time step adaption scheme appeared perfectly reliable in all cases not involving the K-equation.

[4] Compare, however, a remark by Hou & Lowengrub [90]. They have stated that the "singular Biot-Savart kernel in the [3-D vortex-point method] has a natural cut-off", i.e. if vortex elements have initially been separated a distance $h$, they will never come closer than a distance proportional to $h$.

[5] The touching of cores can be defined as the situation in which the distance between two vortons becomes equal to the sum of their core radiuses, defined according to fig.10.2.

### 11.1.3 Correct Representation of Deformation and Interaction

*Vortex Deformation*

All simulations performed with application of the N+K-equation have shown correct representation of deformation (rotation and stretching) of vorticity as long as the vorton rings do not approach closer than a certain distance [6]. Especially the simulations presented in §10.5 have shown that for the N- and K-equation correct deformation is not assured.

*Core Deformation*

We have also observed that the core deforms, which for the elementary case of §10.3 is in accordance with our expectations, at least qualitatively (see fig.10.17 and fig.10.18).

However, from fig.10.17 and fig.10.18 in §10.3 we have found that the core deformation in a vorton ring also depends on the azimuthal position in the vorton ring. For locations between vortons, a rebounding movement of the core has been observed. This behaviour can be regarded as an azimuthal disturbance of the ring, though our numerical simulations did not show any signs of instability [7]. We regard this behaviour as an undesired artefact of the vorton representation. No clear indications exist that the vorton representation really fails at the moment this behaviour sets in.

*Stability*

Regarding the stability of vorton rings (see §10.1.2), we have found rather good agreement with numerical results by Knio & Ghoniem [108], but poor agreement with experimental results; see fig.10.5. An explanation for this weak performance of the vorton method may be the lack of resolution. For the values of the non-dimensional velocity $\tilde{V}$ for which experimental data are available, the number of vortons in the ring is only little more than twice the unstable wave mode number.

Besides, our representation of the vortex ring by just a single "layer" of vortons may be insufficient to adequately represent the internal core dynamics. Knio & Ghoniem have shown that a multi-layer representation of the vortex ring (similar to that shown in fig.10.28) leads to a better agreement with experimental results. An objection against a multi-layered torus is related to the indefiniteness of the positions of the vortons and their initial strengths. Winckelmans [283] has provided pictures which show the presence of unsteady behaviour within the core, which may be due to leap-frogging of the circular vortex filaments which make up the core of the torus. Though Winckelmans's multi-layered ring seems to be stable, the advantages of his representation (see fig.10.28) above a single-layer representation are not clear.

Our simulations of initially distorted head-on colliding vorton rings (see fig.10.23) suggest that a likely explanation for the small-ring formation as found by Lim (see §10.3.2) can be ascribed to the growth of an unstable wave mode on both rings and an ensuing reconnection process. However, full quantitative comparison with Lim's experiment have proved to be impossible (see note 7). An interesting simulation for this configuration would be one of randomly disturbed rings.

[6] One would be tempted to restate this as: as long as cores do not touch. However, evidence for this statement is lacking, since we have not found a indisputable definition of the core size of a vorton ring; see fig.10.3.

[7] For large numbers of vortons (i.e. large than those used in the simulations presented in Chapter 10), we encountered an apparent instability of the vorton rings. This has been the case in our attempt to simulate Lim's configuration of two head-on colliding vortex rings as described in §10.3.2.

*Reconnection*

The simulation of §10.4, which has been performed with the aim of investigating the reconnection of two vortex rings as in the experiment by Izutsu & Oshima (IO), may have bewildered the reader as reconnection is generally supposed to be possible only by viscous annihilation of vorticity (see §C of the Interlude).

Nevertheless, the simulation presented in fig.10.32(c)(ii) shows rather good qualitative agreement with the IO experiment. The development in time does not agree exactly, which may be ascribed to differences in the initial configuration. This is shown, for instance, by the difference in development of the angle of inclination $\theta$ of the rings.

However, our simulation results do not show convincing evidence for the presence of threads (see e.g. fig.10.29) which have been observed in the IO experiment and also in the simulations by Winckelmans [283] and by Kida *et al.* [103]. Fig.10.30 suggests that the formation of threads is related to the "tails" of vorticity which are formed downstream of the vortex rings. These tails may be due to the initial Gaussian distribution of vorticity in the rings:
for equilibrated initial conditions, Gaussian distributions seem not appropriate and the tails may be due to a reorientation of the cores towards an equilibrium shape. This would imply that they are an artefact of the computational modelling [8].

If a dependency of the reconnection process on the Reynolds number really exists, as Anderson & Greengard have suggested (see §10.4.2), then one might wonder whether the interaction of two vortex rings attains a Re-independent behaviour for large Re and whether our simulation is a representation of this limit case. We suggest that the behaviour shown in fig.10.35 only mimics physical vortex reconnection and is just a consequence of the computational model. A clue to this last statement can be found in the results presented in fig.10.32 and mentioned at the end of §10.4.2. The interaction of two vortex rings at the moment they "touch" depends on the arrangement of the vortons in the rings relative to the point of closest approach [9]. A slight disturbance in the symmetry of the configuration may seriously disturb the "reconnection". Another clue can be found in the simulations discussed in §10.2 (fig.10.8), which have revealed that vorton reconnection does not always occur when experiments suggest it should (e.g. in case of axis ratio $L/(2R_e) \approx 7$).

We see that three important questions arise:

1. Is the reconnection observed in our numerical simulations an adequate representation of the physical process?

2. How can reconnection occur in an inviscid simulation?

3. Why does reconnection occur in case of two obliquely interacting vorton rings and not in case of a pseudo-elliptical vorton ring?

Our answer to the first question has already become clear from the remarks above. However, it can only be answered conscientiously if an extensive comparison is made between the numerical and the experimental results. Unfortunately, the latter are still scarce. Besides, the

[8] See the discussion on models in the Epilogue.

[9] Regarding the position of the vortons, we could wonder whether the "dipole" seen in figs.10.32(i) is a numerical artefact only. The "dipole" does not seem to influence the reconnection of the rings. However, we have seen that it does influence the subsequent behaviour of the reconnected vortex ring, i.e. the presence or absence of splitting into two rings.

vorton method may not be able to allow complete simulation on all relevant scales and reveal the exact mechanisms of this phenomenon [10].

Regarding the second question, we first of all have to remark that no real evidence exists for the impossibility of inviscid reconnection [11]. The rejection of inviscid reconnection by some seems to be based [12] on Helmholtz's First Theorem and its interpretation that vortex lines cannot end inside any volume. As remarked in note 3 of Chapter 2, this result is only true for vortex tubes.

Pedrizetti ([177] and [178]) has suggested that the vorton method implicitly introduces a viscous effect during rapid stretching of the vortons. The "viscosity" in this case is proportional to the rate of stretching and the core size. As the vortons involved in reconnection are strongly stretched, the "viscosity" tends to large values and reconnection can happen. According to Pedrizzetti this is "the mechanics which permits to jump over the moment of local intense stretching as vortex reconnection, which, otherwise, could hardly be followed numerically" [177].

In our opinion, this explanation is dubious. We think that the reconnection of two inclined rings can be explained from the "alignment" behaviour of vortons. To illustrate our interpretation of the apparent reconnection of vorton rings, regard the configuration illustrated in fig.11.1. The two pairs of vortons can be imagined to be each part of a vorton ring as in the configuration of §10.4 (the rings are suggested by means of the dotted lines). When the "rings" approach each other and get deformed, the angle between strength vectors of vortons 1 and 2 (indicated by the arrows) changes and their alignment is weakened. At the same time the angle between vortons 1 and 3 changes and their alignment improves. At a certain moment vorton 1 becomes stably aligned with vorton 3 and the "reconnection" has taken place [13].

The above consideration may also settle the question 3 mentioned above. In the case of the pseudo-elliptical vorton ring, the angle between the vortons 1 and 2 does not reach a critical value at which realignment of vorton 1 with vorton 3 is possible [14].

If annihilation of vorticity due to viscosity is really an important ingredient of the reconnection process, we must seriously doubt the applicability of the vorton method to simulate vortex reconnection. A remedy in this case may be the introduction of a viscous term to the vorton equations as has been proposed by Winckelmans [283] (see §10.4.2).

Several authors (e.g. Lim [129]) have pointed at the appearance and importance of helical vortex lines during vortex reconnection. Since vortex lines have not been visualized in our simulations, we cannot tell whether helical vortex lines have been present. Possibly, this twisting of vortex lines can only be simulated correctly if the vortex ring is represented by the multi-layered torus used by Knio & Ghoniem and by Winckelmans (see above).

[10]In future numerical investigations of vortex reconnection, one should examine the behaviour of other diagnostics: besides the isosurfaces of vorticity, those of the rate of strain and enstrophy production may be used for comparison with experimental data. The visualization of vortex lines may also be informative on the exact mechanism of reconnection. However, it can also be misleading, as Robinson has remarked [196].

[11]One surprising result in this regard is the suggestion by Melander & Hussain in [160] that reconnection occurs on a convective timescale.

[12]See e.g. the quotation from [103] given in §10.4.2.

[13]One could call this a bifurcation, due to the resemblance with this mathematical concept.

[14]As remarked in §10.2.2, another possible explanation may be related to the initial restriction (see fig.10.12), though even in that case no reconnection has been found. Another explanation may be related to the fact that in the vorton representation inertia is not included. The inertia of the approaching parts of the ring after the switch of the axes may be responsible for the close approach which subsequently leads to reconnection.

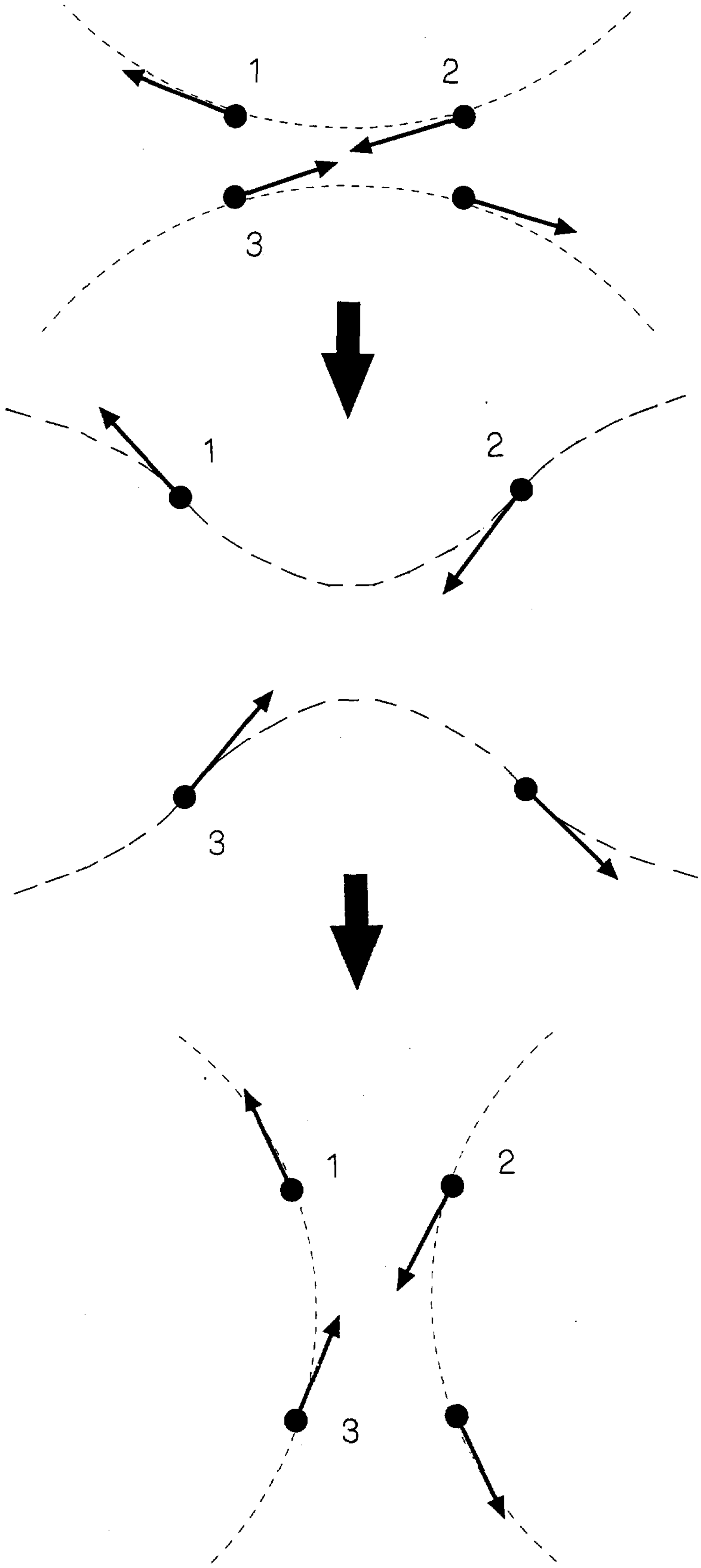



Figure 11.1: Elementary vorton configuration to demonstrate the possibility of "reconnection" (see text). The dotted lines represent parts of vorton rings. Big arrows indicate time development.

*Alignment*

The simulations in §10.5 of two knotted rings have made clear that the vorton rings correctly represent the tendency of alignment. Shortly after the completion of the alignment, however, the simulation breaks down. Since experimental data on this configuration are lacking, it is impossible to make any further remarks or to draw definite conclusions on the physical correctness of the alignment and on the behaviour following alignment. For instance, we would like to know whether reconnection may be expected in this case. We have implemented several initial positions of the vortons in the rings, but in none of these cases reconnection appeared to be (even dimly) present. Possibly, in real viscous flows the aligned anti-parallel vortex tubes will annihilate each other. In case this is an essential part of this interaction, we have to conclude that the application of the vorton method as presented in this thesis is not warranted.

### 11.1.4 Conservation of Motion-invariants

In §9.3.1 and Appendix A it is made clear that our expressions used as diagnostics for the simulations can be criticized. This has been a serious obstacle in drawing any conclusions with regard to the question: do vorton simulations show conservation of the relevant motion-invariants?

The simulations in §10.2 (fig.10.9 and fig.10.10) show that the (nonzero) motion-invariants are rather well conserved as long as no severe vortex (core) deformation takes place. However, fig.10.40 in §10.5 suggests that conservation is seriously violated the moment the cores "touch" and severe stretching and deformation takes place. However, fig.10.33 in §10.4 proofs that reconnection does not necessarily mean a strong violation of conservation.

The time-development of interaction-energy $E_i$ and self-energy $E_0$ have opposite trends (see fig.10.22 and fig.10.33), which suggests that $E_i + \alpha E_0$ (where $\alpha$ is some constant) may be a correct representation of the total kinetic energy.

The simulation in §10.5 (fig.10.40(c)) shows that interaction-helicity $H_i$ is well conserved up to the moment the aligned parts of the vorton rings approach each other closely. The subsequent violation of conservation of $H_i$ may be attributed to the failure of the vorton method to represent annihilation of vorticity, as discussed above.

### 11.1.5 No Negative Effects of Remeshing (Vorton Division)

As remarked in §11.1.2, vorton division may be essential, since it counteracts the growth of cores of vorton rings during stretching. Besides, addition of vortons increases the resolution in areas where this seems essential for correct simulation. However, as explained in §10.3.3, it may engender undesired effects. Besides, the insertion of vortons leads to an abrupt change in the core size and structure. This may be of serious consequence to the stability of the vortex structure. Therefore, inclusion of vorton division should be implemented very carefully [15]. At least, we can conclude from fig.10.24 in §10.3 that division without updating (see §9.4) has to be rejected. Division with updating improves conservation of interaction-energy.

Some additional remarks have to be made. Division will become troublesome in case of reconnection, due to the possibility of a sudden exchange of neighbours between the vortons. The value of factor $\lambda$ (see §9.4) has been derived under the assumption of a circular core hence for strong core deformation, this value may have to be adapted. Furthermore, updating

[15] E.g., in the interpolation of the locations of the added vortons (see fig.9.3) use could be made of spline interpolation. For a radially growing vorton ring, as in §10.3, this will assure a conservation of its circularity.

with regard to circulation $\Gamma$, as we have done (see §9.4), does not imply conservation of other invariants.

The inability of our vorton method to represent annihilation (see above) also causes the final breakdown of the situation illustrated by fig.10.36, i.e. one of the possible effects of vorton division. If annihilation would take place in this situation, the simulations to which we have applied vorton division might show correct development of the simulations of §10.4 and §10.5, instead of breakdown.

### 11.1.6 Correct Boundary Conditions

As explained in §9.1, we have chosen for vorton configurations for which it is not necessary to implement explicitly boundary conditions. Only in case of the simulation presented in §10.6 (simulation of a free-slip flat boundary by means of mirrorimaging), this requirement may be important (see §11.2 below for further discussion).

### 11.1.7 Convergence

In literature, the convergence requirement has been formulated for vortex-point methods by the question [16]: do vortex configurations represented by vortex elements tend to represent continuous vortex structures better and better for an increasing number of vortex elements?

With regard to the vorton method, this question seems irrelevant to us. In §10.1.1 we have shown that increasing the number of vortons implies changing the characteristics of a vorton ring and renders an investigation into convergence, as defined above, impossible [17].

The formulation of the convergence requirement as used in literature and stated above can be replaced by an alternative one. Actually, for a vortex method like the vorton method, we would like to have a proof of a property which has been given for the 2-D point-vortex method. This property is related to the following question: do 2-D vortex points show the same (qualitative) dynamics as patches of vorticity [18] whose behaviour is obtained by directly solving the Euler (Helmholtz) equation? This has been shown both analytically by Marchioro & Pulvirenti [141] and numerically by Benzi *et al.* [23]. For the vorton method, such a proof does not seem feasible in the same way since 3-D patches cannot exist on their own. Instead,

[16] Note that we give only an informally expressed version of this question here. An exact mathematical formulation can be found in literature, e.g. [90].

[17] Despite this situation with regard to the vorton method, several authors have suggested that their proofs of convergence apply to this point-vortex method. However, partly because of their mathematical nature, it is hard to find out whether the results indeed apply to the (soft-)vorton method.

For the soft-vorton method (see Appendix B) convergence seems to have been investigated first by Cottet [41], who proved that the appropriate error norm for the velocity and vorticity fields goes to zero as the number of soft vortons increases and the core-size decreases subjected to the constraint that the cores overlap (i.e. the core sizes have to be larger than the typical distances between the elements). Another proof of convergence for this case has been given by Beale [21]. Winckelmans [283] has shown convergence for the soft-vorton method by means of his numerical simulations, though it appeared to be slow.

Cottet [42] has also shown that his vortex method discussed in [41] converges even without smoothing, thereby apparently providing a proof of convergence for the vorton method. However, the proof required two mathematical tools whose applicability with regard to the vorton method are unclear. Hou in [10] has remarked that the result found by Cottet does not mean that the vortex-point method can be applied without smoothing or desingularization. According to Hou, for any given time $T$ a condition exists for which the method is stable and convergent. However, the number of particles is finite, so there will be a time beyond which particles are so close that stability analysis breaks down. Beyond this time, some "regularization" (i.e. remeshing) is needed.

We have to conclude that a proof of convergence for the vorton method still seems to be absent. However, carefully performed numerical simulations may give valuable clues with regard to this issue.

[18] By patches we mean compact distributions of vorticity.

a close comparison between the dynamics of a vorton ring and a full numerical simulation of a vortex ring will be necessary to serve the same purpose.

### 11.1.8 Computational Effort

In general, we can remark that computational times for our simulations have been satisfactory (in the order of minutes). However, we must add that these simulations have only been done for relatively small numbers of vortons. The relation between computational times and the number of vortons $N$ has been investigated for the simulations discussed in §10.4. We have found that times are proportional to about $N^2$.

Simulations of multi-layered rings like those performed by Winckelmans on the configuration shown in fig.10.28 require a much larger effort [19]. Kascic [101] has suggested the use of a vector processor to simulate the dynamics of large numbers of vortons.

## 11.2 The Vorton Method and research on Coherent Structures

The results of the simulation on the single vorton ring in a shear flow above a flat plate, as presented in §10.6, may be too elementary to allow any conclusions with regard to the applicability of the vorton method to the study of CS in turbulent boundary layers (TBL).

First of all, as indicated in §10.6 experimental evidence for the existence and role of vortex rings or related vortex structures in the TBL is scarce. Furthermore, we do not know whether phenomena like vortex reconnection and annihilation of vorticity are (crucially) involved in the behaviour of any such structures. If this would turn out to be the case, we have to realize that our vorton simulations have shown the inadequacy of the vorton method on this point.

One may also object that the no-slip boundary condition at the surface of the wall and the related generation of secondary vorticity may be crucial for the flow phenomena observed in the wall region of the TBL as some authors have suggested (see §10.6.1).

All the same, our results may illustrate that even elementary and crude configurations can contribute to an understanding of TBL flows [20]. At least, our simulation have shown that an outer region parameter (*in casu* the outer layer velocity $U$) of the shear flow determines the behaviour of the vorton ring and (consequently) the Reynolds stress pattern in the flow. This suggests that outer layer parameters in the TBL (partly) determine its characteristics.

## 11.3 Final Remarks

We think that the simulations presented here have given some indication of the applicability of our vorton method (i.e. applying the N+K-equation). We conclude that the vorton method produces simulation results which agree fairly well with experimental and analytical results, at least when vortex structures, like vortex rings, do not approach each other closer than a certain distance. When vorton structures approach more closely (and e.g. viscous effects are likely to become involved), we have to be very careful in judging the numerical results. Especially the simulations in §10.3 and §10.5 have shown that vorton behaviour may start to become chaotical . However, lack of experimental results prevents more decisive conclusions.

The vorton method may be extended to improve its performance. The use of soft-vortons, of multi-layered vorton rings and the addition of viscous diffusion to the vorton equations are possibilities. However, the first option shows important disadvantages and has nowhere been shown to perform better than the ordinary vorton method (see also Appendix B). For

[19] Winckelmans did not provide details on his computational times; the numbers of vortons he typically used were of the order $10^3 - 10^4$.

[20] For a discussion of the nature and use of modelling in turbulence, we refer to the Epilogue.

the second extension, the correct implementation of such rings is still unclear and simulations require a large computational effort. As for the third option, we only have the results by Winckelmans [283]; the same kind of objections exist as for the second option. Besides, a more careful incorporation of vorton division than that applied in our simulations is necessary to increase the applicability of the vorton method.

The vorton method is a relatively cheap, quick, and simple to handle vortex method, able to provide a first indication of the behaviour of vortex configurations [21]. However, for a really careful simulation of closely interacting vortex structures, the method may not be reliable and the application of a viscous vorton method (like that of Winckelmans) may be more appropriate. Besides, still other numerical (vortex) methods unrelated to the vorton method may be better suited for certain simulations [22].

[21] We agree with Chorin's remark: "a good guess at the solution of the problem one wants to solve is better than an unambiguous solution of the wrong problem" [37].

[22] One recent promising method is that by Verzicco and co-workers, who solve the Navier-Stokes equation by means of a finite-difference scheme (see e.g. [273]).

# Epilogue

In this final chapter, I will attempt to bring together the vortex-atom-part and the vorton-part on a scientific-philosophical level. In the preceding chapters I have shown how both parts are related on the scientific level, e.g. by showing how the theorems and equations first proposed by Helmholtz and Kelvin can be applied for the derivation of the vorton equations, Besides, in the Interlude I have indicated (though only in a superficial manner) how certain aspects of vorticity theory show a continuous development from the days of the vortex atom to the present.

On the scientific-philosophical level, the vortex atom and the vorton are not related in such a direct sense, though at least I will quote some of the 19th century authors who have been mentioned in the vortex-atom-part [23]. The relation I would like to discuss is based on a common and important concept involved in both the vortex-atom theory and the vorton theory: the **model**.

Everyone familiar with any part of science will have some notion when reading this term. Even restricting the discussion to models in physics, it appears difficult to formulate an unambiguous description of this term. I define a model as a representation of a physical concept [24] that is still unknown in details, but of which one has some image. The model is a simplification of reality, but tries to catch the essential aspects of the real concept. This description rouses questions with regard to the meaning of "reality". Here, I will equate reality to experimental observations.

Several kinds of models may be discerned [25]:

- **analytical models**

  These models consist of (sets of) equations which are supposed to describe in mathematical terms the physical concept which has to be modelled. They do not necessarily contain any viewpoints on the physical backgrounds of the concept.

  Examples are Maxwell's famous equations describing electrodynamical phenomena (see §6.3) and Saffman's model of reconnection (shortly mentioned in §C of the Interlude).

- **physical models**

  A physical model is supposed to be a direct representation of some aspects of the physics involved in the concept to be modelled.

  Example of physical models are the vortex atom model as proposed by Kelvin and the model of a coherent structure (CS) in a turbulent boundary layer as presented in §10.6. Both examples will be fully discussed in this Epilogue.

- **conceptual models**

  In the case of conceptual models one does not suppose that the ingredients used in the model necessarily form a real physical representation of the concept to be modelled (in

[23] More often than is usual nowadays, these scientists occasionally discussed the philosophical backgrounds of their own and others' research.

[24] By physical concept I mean anything which physicists tend to model: objects, phenomena, processes, etc.

[25] Here, as everywhere else in this Epilogue, I will use my own terminology. This list is not exhaustive.

contrast with physical models). The conceptual model and reality have to show similar properties, but the model does not necessarily represent the physical background of the concept.

As an example of this type of models I mention the mechanical (or mechanistic) models favoured by British scientists (e.g. Kelvin) in the second half of the 19th century (see the introduction of Chapter 3 and of Chapter 5). They essentially amounted to the representation of physical concepts by means of a "mechanism" involving springs, wheels, gyroscopes, etc. The laws of (classical) mechanics determined then the behaviour of these models [26].

The aim of a model is to aid in the visualization (and possibly quantification) of physical phenomena and in the understanding of the physical "mechanisms" which determine the character of a physical concept. Some remarks on models can be found in one of the first essays on the use of models, i.e. Rankine's discussion [185] of his own conceptual model of matter, the molecular vortices (see §3.1). According to Rankine a model (or "hypothesis" as he called it)

> substitutes a supposed for a real phenomenon, ... the object being to deduce the laws of the real phenomenon from those of the supposed one. If the supposed phenomenon were more complex, or less completely known in its laws than the real one, the hypothesis would be an incumbrance, and worse than useless. ...
>
> A hypothesis is absolutely disproved by any facts that are inconsistent with it. ... On the other hand, no hypothesis is capable of absolute proof by any amount of agreement between its results and those of observation; such agreement can give at best only a high degree of probability to the hypothesis. ...
>
> The agreement should be mathematically exact, to that degree of precision which the uncertainty of experimental data renders possible, and should be tested in particular cases by numerical calculation. The highest degree of probability is attained when a hypothesis leads to the prediction of laws, phenomena, and numerical results which are afterwards verified by experiment. [185, p.127]

Though Rankine regarded his own hypothesis of molecular vortices as respecting these rules, he warned that hypotheses like these "never can attain the certainty of observed facts" [185, p.132].

On the final fate of models we can read in Larmor's address to the section of Mathematical and Physical Science at the 1900 meeting of the British Association: "When a physical model of concealed dynamical processes has served this kind of purpose ..., when its content has been explored and estimated, and has become familiar through the introduction of new terms and ideas, then the ladder by which we have ascended may be kicked away, and the scheme of relations which the model embodied can stand forth in severely abstract form" [117, p.626].

Naturally, a single phenomenon may be represented by several models or kinds of models. This has been the situation in British science in the latter half of the 19th century. As Duhem remarked in [51] (see §5.2), the British proposed one model for one group of laws and another

[26] Another example may be the model of turbulence proposed by Synge & Lin (see §B of the Interlude) in which the interaction of vortices is supposed to provide characteristics similar to those of turbulent flows. However, it is not clear whether they meant this as a conceptual or as a physical model.

completely different model for another group, though both groups contained some common laws [27].

For Kelvin, around the time of his 1884 Baltimore lecures, the use of models (usually of conceptual nature) was of fundamental importance: "It seems to me that the test of 'Do we or not understand a particular subject in physics?' is, 'Can we make a mechanical model of it?'" [99, p.111] and: "I never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand" [99, p.206]. However, he also stated that his models by no means reflected reality. They were only imitations of reality and certainly not unique.

To further explore Kelvin's use of models and to illustrate the problems related to models, let me now concentrate on his vortex atom model. It is important to notice that the vortex atom theory actually involves two kinds of modelling.

Whereas for his mechanical models Kelvin did not claim a reflection of reality, with regard to the relation between vortex atoms and matter, Kelvin suggested that the *physical* vortex ring, as seen in Tait's experiment, was a physical model of the atom. To confirm this opinion, I refer to the account of Kelvin's 1867 lecture "On Vortex Atoms" [243] (see §4.2), where we read:

> After noticing Helmholtz's admirable discovery of the law of vortex motion in a perfect liquid ... the author [=Kelvin] said that this discovery inevitably suggests the idea that Helmholtz's rings are the only true atoms. [243, p.1]

This suggests that initially Kelvin indeed regarded the vortex atom as a physical representation of the atom. However, I think that many of his contemporaries could only regard it as a conceptual model, as is expressed by Larmor in the quotation given at the end of §6.3.

In order to demonstrate the correctness or usefulness of his (physical) model, Kelvin had to show that it possessed the properties of "real" atoms. That is to say, the properties which were known at that time. As he must have realized that this would be difficult to show experimentally and unconvincing, he went for the analytical elaboration of another type of model of the vortex ring itself. However, according to the definition given above this model cannot be called an analytical model and I shall name it a **computational model**. For Kelvin, initially *the* computational model of the vortex ring was the Kelvin-ring (see §A.2 of the Interlude).

Regarding the story of the vortex atom, I conclude that for Kelvin and his contemporaries, the vortex atom failed as a physical model, since it appeared to lack some of the fundamental properties related to real atoms (stability, gravity, spectra; see §5.3). However, regarding the development of the vortex atom from the present point of view, one can claim that the vortex atom suffered from another weakness. Today, we know that the Kelvin-ring is only a crude computational model of a vortex ring (see §A.2 of the Interlude). At that time, however, this

[27]Duhem stressed that it is better to have one unique theory since this provided the "classification naturelles des lois" and showed the order in nature. On the other hand, he argued that such a situation as in British physics had to be allowed: "Si l'on astreint a n'invoquer que des raisons de logique pure, on ne peut empêcher un physicien de représenter par plusieurs théories inconciliables soit des ensembles divers de lois, soit même un groupe unique de lois; on ne peut condamner l'incohérence dans le développement de la theorie physique" [51, p.366].

circumstance did *not* contribute to the fall of the vortex atom since this computational model was generally regarded sufficient. If, for example, Kelvin would have applied the analytical techniques of Widnall and co-workers (see §A.3 of the Interlude), he could have demonstrated the inherent instability of the Kelvin-ring and would have been forced to revise his computational model. Kelvin's eventual recognition of this weakness can be deduced from his 1889 remark on Hicks's hollow vortex (see §6.3). However, by then the vortex atom had already appeared unviable as a *physical* model.

The story of the vortex atom shows us other factors which may influence the viability of any model [28]:

- In general, one can say that in the case of the vortex atom there has been no rational line of defense. The elaboration of the model (both with regard to its role as physical and as computational model) lacked a real research program. Besides, no attempts were made to refute fundamental criticism, such as that by Reynolds (see §5.1).

- The promoters of the vortex atom theory did not provide quantitative data, which meant that comparison with experimental data was impossible. If more fundamental experiments would have been done on the properties of vortex rings (e.g. on their velocity, distribution of vorticity in the core), the computational part of the model would have been discredited. If quantitative comparison with properties of matter (e.g. spectral lines; see Julius's contribution discussed in §5.3.4) would have been performed seriously, the physical part of the model would have been discredited.

- Relatedly, the model was not predictive (see Rankine's remarks quoted above). Therefore, the model must have appeared useless and irrelevant and could not show any advantage above other models.

- The nature of the model was purely hydrodynamical, while experiments eventually showed that other aspects (e.g. electrical) were essential for an atom model. Therefore, it is not surprising that the vortex atom did not survive the discovery of the electron and of radioactivity [29]. Attempts to adapt the vortex atom model (e.g. by introducing hollow cores; see §5.1), or to extend it (e.g. with electric charge; see §6.2) could not prevent its fall.

- The computational model was hard to elaborate due to lack of sufficient mathematical techniques. Though this was fully recognized by the promoters of the theory and though they introduced several new techniques, progress was slow and several important issues could not be tackled properly.

However, the vortex atom model did not only fail as a result of internal inconsistencies. It also suffered from the shift in the use of models which occurred at the end of the 19th century in British physics (see §6.3). When the vortex atom had started to decline, Kelvin complained that Maxwell's use of analytical models had superseded his own use of physical and conceptual models. Maxwell's initial emphasis on analogy and heuristic models (e.g. his molecular vortices; see §5.1) had changed towards an approach according to which physical phenomena were framed into mathematical equations, i.e. the use of analytical models. To

[28] The order of these factors does not indicate their relative importance.

[29] Likewise the vortex ether did not survive relativity.

Kelvin Maxwell's equations were examplary for the wrong approach. They were "metaphysical" and had been worked out in the mind without contemplation of physical reality. Today, we can conclude that Maxwell's analytical model has been much more successful than Kelvin's physical and conceptual models.

Taking into account the factors mentioned above, it may seem remarkable that the vortex atom model *could* survive for almost 30 years. However, this can be attributed to several favourable circumstances. First, competing theories of matter lacked the same or other fundamental and practical problems. Secondly, lack of experimental data on the properties of matter prevented a definite judgement. Thirdly, the crucial role of electric charge in matter became only fully realized after J.J. Thomson's 1897 discovery of the electron. And last, but not least, Kelvin's fame must have played some role here.

In conclusion, one can say that many factors are involved in the development of a model. Some are evident, others are not. Some can be analyzed rationally, for others this seems impossible. I think that all factors mentioned above, both favourable and unfavourable to the viability of models, can still be found today. Some evidence for this statement can be found in the next and last part of this Epilogue.

In this last part, I will treat two analogies I have found between the vortex-atom-part and the vorton-part with regard to the use of models:

- computational modelling:

  Kelvin ring $\leftrightarrow$ vortex ring $\sim$ vorton ring $\leftrightarrow$ vortex ring

- physical modelling:

  vortex atom theory $\leftrightarrow$ matter $\sim$ vortex models of a CS $\leftrightarrow$ turbulence

As mentioned above, one of the obstructing factors in the development of the vortex atom model has been the lack of proper mathematical techniques. In modern fluid dynamics research the use of numerical techniques has proven to be an important and fruitful new tool to elaborate models. Vortex methods (see Chapter 7) form one part in this field of so-called computational fluid dynamics. Despite the progress in computational capabilities provided by these methods, one still needs a computational model of vortex structures, e.g. vortex rings, on which to apply the numerical tools. In our investigation of the vorton method, the computational model of the vortex ring has been the vorton ring as illustrated in fig.9.1. In Chapter 10 I have investigated the correctness of this model by means of numerical simulations. From the discussion in Chapter 11 the reader may have deduced that the usefulness of the vorton ring as a computational model can be questioned [30]. Moreover, as has been the case for the vortex atom, a good comparison of numerical with experimental results may be impossible, not only due to the scarcity of experiments but also due to the fact that viscosity may have an essential influence on experimental rings.

The second analogy, related to physical modelling, can be found in the vorton simulation treated in §10.6. There, I discussed the present trend in fluid mechanics to regard the role of coherent structures in turbulent flows, *in casu* turbulent boundary layer flows. As discussed in §10.6.1, some have suggested that these structures can be modelled as vortex rings. Notice that this physical modelling is different from that in case of the vortex atom. The vortex

[30] For large numbers of vortons in the vorton ring, it even approaches the Kelvin-ring and we may expect the same problems of modelling as in case of the vortex atom.

ring, Kelvin supposed, could be completely *identified* with the atom. Today, the vortex ring is regarded as an essential part of turbulent flows and is not identified with anything else; it is just a vortex ring.

Turbulence modelling has a history showing important shifts in approach (see also §B of the Interlude). After Kelvin's and FitzGerald's 1887 vortex ring model, at the beginning of this century modelling of turbulence had become largely analytical. In the 1930s the statistical approach began to dominate research in turbulence, which lacked the use of models. Only in the 1950s one realized again the importance of modelling and the concept of "coherent structures" was introduced. Nowadays, research on CS shows a large variety of (vortical) structures which are proposed as explanation for physical phenomena in turbulent flows.

Besides these typically physical models, it should be mentioned that today several other types of modelling are used in turbulence research. This can only be encouraged, as Duhem already realized (see above). Lumley in [135] commented on the ability of turbulence models (including statistical methods) to increase our understanding of turbulence: "However, I believe it is foolhardy to expect them to. These models are simply embodyment of experience; they are something constructed to behave like turbulence, in situations where it has been observed, to be used as a design tool. A model cannot, except by accident, contain more than is put into it."

Naturally, this last remark is relevant to any kind of model. Because electric charge had not been put into the vortex atom model, it was unable to model the atom. In our present models of turbulent flows, we should strive for models which can surpass, so to speak, limits. The main problem will be how to set up such models and how to interpretate their results. Some models may suggest "too much" but this is not important; the point lies in suggestion, not demonstration [31]. Nevertheless, one has to be alert that models may become more important than the phenomena themselves.

We can only hope that the models of CS will improve our understanding of turbulence as a physical phenomenon. The central question in this respect has been formulated by Kline & Robinson in [73] as: "how do we capture the essence of such a model in a simple enough way so that it becomes useful in creating predictive models?" We must realize that even negative results can help us and that the road towards complete understanding, if ever achieved, certainly isn't straight.

As in the case of the vortex atom, the most important problem here is the relatively small amount of knowledge on the characteristics of CS and their role in the TBL from experiments. We even have a lack of definition and problem formulation (as already remarked in §B of the Interlude). As long as this situation lasts, models cannot be judged correctly.

Though of enormous help in advancing our knowledge on fluid flows, the present trend of computational fluid mechanics brings along its own problems. For the specific case of vortex methods, the computational modelling of vortices (as discussed above for the vorton method) should be performed very carefully. In addition to this, other problems arise due to the large amount of data provided by numerical simulations. The problem of selecting those data which are useful for the purpose of understanding will only become more urgent. And, relatedly, we have to face the problem also formulated by Kline & Robinson in [73]: "how do we combine the results from numerical simulation data bases with experimental results to approach consensus on a complete model of structure [in turbulence]?" Another problem related to computational methods are the possible influences of numerical artefacts (e.g. numerical viscosity). These

[31] Compare the suggestion of the vortex atom to J.J. Thomson in his discovery of the electron (see §6.3).

may disturb our view on the real physical value of models.

One may think differently about the fate of models, but surely the energy put into their elaboration will not be lost, even if the picture they provide does not correspond to reality or only slightly. Models can give impulses towards new developments and the mathematical topics which they induce may well be worth treatment themselves. Though the vortex atom model itself failed, it left behind a heritage: it meant an important stimulus to the research on vortex motion and led to Tait's contribution and foundation of the theory of knots (see §C of the Interlude).

To conclude, I remark that a model must be used as a first step in investigating the physics of a phenomenon. Afterwards, experimental and analytical results have to be invoked to lay down a theory. At that time, Larmor's "ladder" may be kicked away. Kelvin already realized the relative value of models and eventually left the vortex atom for new models in which he inserted new concepts arising in physics. Perhaps, one day, we have to recognize that our present approach to the modelling of turbulence is unfruitful or should be improved. Then, we must dare to shift towards new approaches.

# Appendix A

# Vector Potentials and Motion-Invariants

From an elaboration of an *divergence-free* vector potential field, conclusions can be drawn regarding its highest order term and its relation with the conservation of motion-invariants [1].

It is assumed that the vorticity field $\boldsymbol{w}(\boldsymbol{x})$ decays fast enough:

$$\frac{wL}{U} \sim [\frac{L}{x}]^N \text{ as } \frac{x}{L} \to \infty \tag{A.1}$$

where $w \equiv |\boldsymbol{w}|$, $L$ is a typical length scale, and $U$ is a typical velocity scale. This condition is certainly fulfilled if vorticity decays exponentially [2].

For a divergence-free $\boldsymbol{A}$ we have the Poisson equation:

$$\nabla^2 \boldsymbol{A} = -\boldsymbol{w} .$$

For the far field condition for the velocity field $\boldsymbol{v}$

$$v \to 0 \text{ as } x \to \infty$$

we have the solution

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi} \int_{V'} \frac{\boldsymbol{w}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

where $V'$ is the vorticity-containing volume. By expanding this integral in powers of $r^{-1}$ ($r \equiv |\boldsymbol{x}|$), we get:

$$\boldsymbol{A} = \sum_{n=0}^{m} \boldsymbol{A}^{(n)} + O(r^{-m-2})$$

where

$$\boldsymbol{A}^{(n)} = \frac{1}{4\pi r^{n+1}} \int_{V'} \boldsymbol{w}(\boldsymbol{x}')[(r')^n P_n(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}}')]$$

where $\hat{\boldsymbol{x}} \equiv \boldsymbol{x}/r$ and $P_n$ are Legendre functions.

The first three terms of this sum are given by:

$$\begin{aligned}
\boldsymbol{A}^{(0)} &= \frac{1}{4\pi}\frac{1}{r} \int_{V'} \boldsymbol{w} \\
\boldsymbol{A}^{(1)} &= \frac{1}{4\pi}\frac{\partial}{\partial x_i}(\frac{-1}{r}) \int_{V'} x_i \boldsymbol{w} \\
\boldsymbol{A}^{(2)} &= \frac{1}{8\pi}\frac{\partial}{\partial x_i x_j}(\frac{1}{r}) \int_{V'} x_i x_j \boldsymbol{w} .
\end{aligned}$$

[1] This disucssion is based on [264]; see also [205, §3.2].

[2] Compare this condition with that given in (9.4).

The (far field) description of the vector potential $\boldsymbol{A}$ to the order $r^{-m-1}$ is defined by the $n$th moments of vorticity for $n \leq m$. These moments of vorticity are defined by:

$$\int_V \boldsymbol{w} \prod_{i=1}^{3} x_i^{j_i} \quad \text{with } j_i \geq 0 \text{ and } \sum_{i=1}^{3} j_i = n \, . \tag{A.2}$$

All $n$th moments exist for any $n \leq N-3$, where $N$ is the order of the far vorticity field defined in (A.1).

Using this condition (A.1), it can be shown that an $n$th coaxial moment of a divergence-free vorticity field $\boldsymbol{w}$ along an axis parallel to any vector $\boldsymbol{b}$ should vanish [264, §1.2], i.e.:

$$I^{(n)}(t, \boldsymbol{b}) \equiv \int_V (\boldsymbol{x} \cdot \boldsymbol{b})^n \boldsymbol{w} \cdot \boldsymbol{b} = 0$$

for $t \geq 0, n = 0, 1, 2, \ldots$.

From this result, we can derive consistency conditions, which are linear combinations of the $n$th moments of vorticity:

- For $n = 0$, we have the consistency condition:

$$\boldsymbol{\Omega} \equiv \int_V \boldsymbol{w} = 0 \, .$$

  This result expresses conservation of the total vorticity. It also shows that the highest order term of the vector potential $\boldsymbol{A}$ disappears, i.e.

$$\boldsymbol{A}^{(0)} = 0$$

  which means that the highest order term of $\boldsymbol{A}$ is $r^{-2}$. Apparently, total vorticity is always zero for a bounded flow on whose surface $\boldsymbol{w} \cdot \boldsymbol{n} = 0$.

- For $n = 1$, we have the consistency condition:

$$\int_V x_i w_j + \int_V x_j w_i = 0 \ \ (i, j = 1, 2, 3 : j \leq i) \, .$$

  From the required far-field behaviour of the fields $\boldsymbol{v}$ and $\boldsymbol{w}$, it follows that the first moment of vorticity (see (A.2)) is time invariant, i.e.

$$\frac{\partial}{\partial t} \int_V x_i \boldsymbol{w} = 0 \, .$$

  Using the above consistency condition, we then derive:

$$\boldsymbol{P} \equiv \int_V \boldsymbol{x} \times \boldsymbol{w} = \boldsymbol{P}_0 \tag{A.3}$$

  where $\boldsymbol{P}_0$ is the constant vector specified by the initial data. This result expresses conservation of the total linear momentum.

- For $n = 2$, we can derive from the consistency conditions:

$$\boldsymbol{J} \equiv \int_V x^2 \boldsymbol{w} = \boldsymbol{J}_0 \tag{A.4}$$

where $\boldsymbol{J}_0$ is the constant vector specified by the initial data. This result expresses conservation of the total angular momentum.

Combining this relation with the consistency conditions related to $\boldsymbol{P}$, we find another expression for motion-invariant $\boldsymbol{J}$:

$$\boldsymbol{J} = \frac{1}{3} \int_V \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{w}) \,. \tag{A.5}$$

- No additional invariants for $n \geq 3$ have been found [264, §1.2].

Now we regard the vorton fields, presented in §8.2. The vector potential field (8.8) chosen in §8.2 is of order $r^{-1}$ and thus violates the condition derived above. However, since this field is not divergence-free, the above discussion isn't necessarily applicable here. On the other hand, for our vorton vector potential (8.8):

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{-1}{4\pi} \sum_{\alpha} \frac{\boldsymbol{R}_\alpha \cdot \boldsymbol{\gamma}_\alpha}{R_\alpha^3} \,.$$

Hence, $\boldsymbol{A}$ may be divergence-free at an infinite number of points. At these points, the above discussion *is* relevant and defence of the choice of our vorton vector potential (8.8) seems untenable.

The condition on the order of a divergence-free vector potential *is* fulfilled for the Chefranov vortex-dipoles (see e.g. [36]; see for a discussion [206]). The vortex dipoles can be regarded as (infinitesimal) vortex rings. Chefranov has claimed that the equations for their dynamics have a Hamiltonian structure and that his method satisfies conservation of linear motion-invariants. However, no numerical simulations seem to have been performed applying this dipole method.

However, these dipoles may not be suitable for numerical simulations as done in Chapter 10, due to their self-velocity [3]. Synge & Lin [225] (see §B of the Interlude) have investigated the interaction of dipoles in their search for a model of turbulence, but concluded it had "undesirable features" and did not lead to correct correlation functions.

[3]Possibly this self-velocity can be eliminated by the addition of swirl, as has been proposed by Moffatt in [162] in the context of his alternative "vorton" model (see e.g. [158]). According to Moffatt, the original vorton is not a useful concept, as it is no solution of the Euler equation (private communication). The vorton should be a "structure of compact support" that propagates with self-induced velocity and without change of structure and can be regarded as a generalization of the vortex ring. Turbulence, he suggested in [135], could perhaps be regarded as a "sea of interacting vortons". Unfortunately, this theory has not been elaborated yet.

# Appendix B

# The Soft-Vorton Method

One of the aspects of the vorton method which has been criticized is the singular behaviour of its velocity and vorticity fields. Therefore, Kuwabara [114] has proposed to replace the delta-functions in the original vorton vorticity field (8.5) by smooth functions [1]:

$$\boldsymbol{w}_\sigma(\boldsymbol{x},t) = \sum_\alpha \boldsymbol{\gamma}_\alpha(t)\,\zeta_\sigma(\boldsymbol{x}-\boldsymbol{r}_\alpha(t)) \tag{B.1}$$

where a choice has to be made for the so-called smoothing function $\zeta_\sigma$ for which we require:

$$\zeta_\sigma(\boldsymbol{x}) \to \delta(\boldsymbol{x}) \ \text{ as } \ \sigma \to 0\,. \tag{B.2}$$

This function contains a parameter $\sigma$, which can be regarded as a "core radius" of the smoothed vortons.

As remarked in §7.3.3, Kuwabara's soft-vorton method is an example of the Smoothed Vortex-Point Methods. A more general treatment of its theory has been provided by Winckelmans [283], who applied this vortex method in several numerical simulations. The simulations by Winckelmans [283] for the configuration of §10.4, have showed similar results for the original vorton method and the soft-vorton method, both with regard to conservation of diagnostics and to reconnection behaviour.

We shortly repeat two drawbacks of the soft-vorton method, which have already been mentioned in the introduction of Chapter 8. First, the field (B.1) is not divergence-free, like the original vorton vorticity field (8.5). Secondly, several smoothing functions $\zeta_\sigma$ can be applied under the imposed requirements, presumably leading to different simulation results.

The first drawback can easily be suppressed as in the case of the vorton method by deriving a divergence-free vorticity field from an appropriate vector potential $\boldsymbol{A}_\sigma$ [2].

We start from (compare with (8.8)):

$$\boldsymbol{A}_\sigma(\boldsymbol{x},t) = \sum_\alpha \phi_\sigma(\boldsymbol{R}_\alpha,t)\,\boldsymbol{\gamma}_\alpha(t)\,.$$

This time, the function $\phi_\sigma$ remains undetermined and is a function of $\sigma$. In the manner shown in §8.2, we derive a velocity field:

$$\begin{aligned}\boldsymbol{v}_\sigma(\boldsymbol{x},t) &= \sum_\alpha \boldsymbol{\nabla}\phi_\sigma \times \boldsymbol{\gamma}_\alpha \\ &= \sum_\alpha (\boldsymbol{\nabla}\phi(\boldsymbol{R}_\alpha)\times\boldsymbol{\gamma}_\alpha) g_\sigma(\boldsymbol{R}_\alpha) \\ &= \frac{1}{4\pi}\sum_\alpha \frac{\boldsymbol{\gamma}_\alpha\times\boldsymbol{R}_\alpha}{R_\alpha^3} g_\sigma(\boldsymbol{R}_\alpha)\,.\end{aligned} \tag{B.3}$$

[1] A similar vortex method has been proposed by Mosher [165].
[2] The $\sigma$ will indicate a soft-vorton function or field.

where $\phi$ is the function defined by (8.9). And from this velocity field we derive the vorticity field:

$$\boldsymbol{w}_\sigma(\boldsymbol{x},t) = \sum_\alpha \{\boldsymbol{\gamma}_\alpha \zeta_\sigma(\boldsymbol{R}_\alpha)) + \boldsymbol{\nabla}(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{\nabla}\phi_\sigma)\} \quad \text{(B.4)}$$

$$= \frac{1}{4\pi}\sum_\alpha \{\boldsymbol{\gamma}_\alpha \zeta_\sigma(\boldsymbol{R}_\alpha) - \boldsymbol{\nabla}[\frac{\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_\alpha}{R_\alpha^3} g_\sigma(\boldsymbol{R}_\alpha)]\} \quad \text{(B.5)}$$

$$= \frac{1}{4\pi}\sum_\alpha \{[\boldsymbol{\gamma}_\alpha - \frac{(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_\alpha)\boldsymbol{R}_\alpha}{R_\alpha^2}]\zeta_\sigma(\boldsymbol{R}_\alpha) - \boldsymbol{\nabla}[\frac{\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_\alpha}{R_\alpha^3}]g_\sigma(\boldsymbol{R}_\alpha)\} \,.$$

In these derivations, use has been made of the following relations between the functions $\bar{\phi}_\sigma(\rho) \equiv \sigma\phi_\sigma(\boldsymbol{x})$, $\bar{g}_\sigma(\rho) \equiv g_\sigma(\boldsymbol{x})$, and $\bar{\zeta}_\sigma(\rho) \equiv \sigma^3\zeta_\sigma(\boldsymbol{x})$ (where $\rho \equiv \boldsymbol{x}/\sigma$):

$$-\nabla^2\bar{\phi}_\sigma(\rho) = \bar{\zeta}_\sigma(\rho)$$

$$\bar{g}_\sigma(\rho) = -\rho^2\bar{\phi}'_\sigma(\rho) \,.$$

If we impose the condition that the function $\bar{\zeta}_\delta$ satisfies the following normalization (convergence) condition:

$$4\pi\int_0^\infty \bar{\zeta}_\delta(\rho)\rho^2 ds = 1 \,,$$

we find:

$$g(\rho) \to 1 \text{ as } \rho \to \infty \,. \quad \text{(B.6)}$$

Consequently the field (B.3) converges towards the vorton velocity field (8.10) for $\sigma \to 0$.

From the soft-vorton fields derived above, the soft-vorton displacement and deformation equations can be derived.

The displacement equation is easily obtained from the velocity field (B.3):

$$\dot{\boldsymbol{r}}_\alpha = \boldsymbol{v}_\sigma(\boldsymbol{r}_\alpha,t) \quad \text{(B.7)}$$

$$= -\sum_\beta (\boldsymbol{\nabla}\phi_\sigma(\boldsymbol{R}_{\alpha\beta}) \times \boldsymbol{\gamma}_\alpha) g_\sigma(R_{\alpha\beta}) \quad \text{(B.8)}$$

$$= \frac{1}{4\pi}\sum_\beta \frac{\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_{\alpha\beta}}{R_{\alpha\beta}^3} g_\sigma(\boldsymbol{R}_{\alpha\beta}) \,. \quad \text{(B.9)}$$

For the derivation of the deformation equation of a soft vorton $\alpha$ we can make use of the splitting of fields mentioned in §8.3 (see also Appendix C): $\boldsymbol{v}_\sigma = \boldsymbol{v}_\sigma^\alpha + \tilde{\boldsymbol{v}}_\sigma^\alpha$ and $\boldsymbol{w}_\sigma = \boldsymbol{w}_\sigma^\alpha + \tilde{\boldsymbol{w}}_\sigma^\alpha$. We then have to elaborate (in case of the ordinary representation of the Helmholtz equation):

$$\frac{D(\boldsymbol{w}_\sigma^\alpha + \tilde{\boldsymbol{w}}_\sigma^\alpha)}{Dt} = ((\boldsymbol{v}_\sigma^\alpha + \tilde{\boldsymbol{v}}_\sigma^\alpha)') \circ (\boldsymbol{w}_\sigma^\alpha + \tilde{\boldsymbol{w}}_\sigma^\alpha) \quad \text{(B.10)}$$

at $\boldsymbol{x} = \boldsymbol{r}_\alpha$.

We will not show this elaboration here and refer to [283] for further details.

# Appendix C

# Derivation of the Vorton Equations

From the soft-vorton displacement equation (B.9) presented in Appendix B, we can easily derive the vorton displacement equation by taking $\sigma \to 0$ and applying (B.6). The terms $\beta = \alpha$ disappear in a natural manner since for these $g_\sigma = 0$. This means that "self-displacement" of a vorton is omitted [1]. The full equation is given by (8.15) in §8.3.

In first instance, one would be inclined to derive the vorton deformation equation in the same manner as we did for the vorton displacement equation, i.e. from the soft-vorton deformation equation. However, we will take a more direct route and apply the vorton fields directly to the Helmholtz equation. However, because of the delta-functions involved, we have to resort to a technique (mentioned in §8.3) by which the Helmholtz equation will be integrated about the sphere $B_\alpha$ of radius $\epsilon$ and centre $\boldsymbol{r}_\alpha$. It is assumed that $\epsilon$ is so small that no other vortons are inside the sphere. We call this a weak formulation [2].

For convenience, we split the fields into two parts as described in §8.3. Thus, we have to calculate:

$$\int_{B_\alpha} \frac{D(\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha)}{Dt} = \int_{B_\alpha} ((\boldsymbol{v}^\alpha + \tilde{\boldsymbol{v}}^\alpha)') \circ (\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha) \; . \tag{C.1}$$

The lefthand side of (C.1) can be rewritten as:

$$\int_{B_\alpha} \frac{D(\boldsymbol{w}^\alpha + \tilde{\boldsymbol{w}}^\alpha)}{Dt} = \int_{B_\alpha} \frac{D\boldsymbol{w}^\alpha}{Dt} = \frac{d}{dt}\int_{B_\alpha} \boldsymbol{w}^\alpha \; .$$

The last integral is not equal to $\boldsymbol{\gamma}_\alpha$, as one might expect. Since $\boldsymbol{w}^\alpha = \boldsymbol{\nabla} \times \boldsymbol{v}^\alpha$, we get:

$$\begin{aligned}
\int_{B_\alpha} \boldsymbol{w}^\alpha &= \int_{B_\alpha} \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi(\boldsymbol{R}_\alpha) \times \boldsymbol{\gamma}_\alpha) \\
&= -\frac{1}{4\pi\epsilon^2}\int_{\partial B_\alpha} \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\gamma}_\alpha) \\
&= -\frac{1}{4\pi\epsilon^2}\int_{\partial B_\alpha} \{(\boldsymbol{n}\cdot\boldsymbol{\gamma}_\alpha)\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{\gamma}_\alpha\} \\
&= -\frac{1}{4\pi\epsilon^3}\int_{B_\alpha} \boldsymbol{\nabla}(\boldsymbol{R}_\alpha \cdot \boldsymbol{\gamma}_\alpha) + \frac{1}{4\pi\epsilon^2}4\pi\epsilon^2\boldsymbol{\gamma}_\alpha \\
&= -\frac{1}{4\pi\epsilon^3}\frac{4}{3}\pi\epsilon^3\boldsymbol{\gamma}_\alpha + \boldsymbol{\gamma}_\alpha
\end{aligned}$$

[1] The exclusion of the term $\beta = \alpha$ can be compared to the so-called cut-off of the kernel of the rule of Biot-Savart (2.3) (see §7.1). A cut-off corresponds, physically, to a finite core size. This suggests that the shortest distance between vortons can be regarded as a core size, a result which has become evident in §10.1 and especially in §10.3.

[2] In our derivation, it is implicitly assumed that the sphere $B_\alpha$ only contains vorton $\alpha$. Consequently, we require $R_{\alpha\beta} > \epsilon$ for all $\beta \neq \alpha$. On the other hand, the value of $\epsilon$ is assumed $\ll 1$. This means that in case of very close approach of vortons the equations will not be reliable. Fortunately, the simulation presented in §10.3 suggests that close approach will not occur.

$$= \frac{2}{3}\boldsymbol{\gamma}_\alpha$$

where $\boldsymbol{n} \equiv \boldsymbol{R}_\alpha / R_\alpha$ and $\partial B_\alpha$ is the surface of sphere $B_\alpha$. Note that this result is independent of $\epsilon$.

Comparing this result with expression (8.11), we derive the equality [3]:

$$\int_{B_\alpha} \phi''(\boldsymbol{R}_\alpha) \circ \boldsymbol{a} = -\frac{1}{3}\boldsymbol{a} \tag{C.2}$$

for any $\boldsymbol{a} \neq \boldsymbol{a}(\boldsymbol{x})$. Apparently, the second (nonlocal) part of the vorton vorticity field reduces the total amount of vorticity inside sphere $B_\alpha$ with a factor of one third.

Next, we will investigate the four parts of the righthand side of (C.1) separately. Use will be made of the following relation:

$$((\boldsymbol{a} \times \boldsymbol{\nabla}\phi(\boldsymbol{x}))') \circ \boldsymbol{b} = \boldsymbol{a} \times (\phi''(\boldsymbol{x}) \circ \boldsymbol{b}) \tag{C.3}$$

for any $\boldsymbol{a} \neq \boldsymbol{a}(\boldsymbol{x})$ and $\boldsymbol{b} \neq \boldsymbol{b}(\boldsymbol{x})$.

1. $\int ((\boldsymbol{v}^\alpha)') \circ \boldsymbol{w}^\alpha$ represents the "self-deformation" of vorton $\alpha$.

   Like self-displacement, this part is omitted. This could be justified, as we did in the derivation of the vorton displacement equation above, by the fact that the all components of matrix $((\boldsymbol{v}^\alpha_\sigma)')$ equal zero for $\boldsymbol{x} = \boldsymbol{r}_\alpha$.

2. $\int ((\tilde{\boldsymbol{v}}^\alpha)') \circ \boldsymbol{w}^\alpha$ represents the deformation of the vorticity field $\boldsymbol{w}^\alpha$ generated by vorton $\alpha$ itself, due to the velocity field $\tilde{\boldsymbol{v}}^\alpha$ generated by all other vortons at the location of vorton $\alpha$.

   Rewriting this integral as

   $$((\tilde{\boldsymbol{v}}^\alpha)') \circ \int \boldsymbol{w}^\alpha ,$$

   we get, by applying (C.3):

   $$-\sum_{\beta \neq \alpha} \boldsymbol{\gamma}_\beta \times (\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \frac{2}{3}\boldsymbol{\gamma}_\alpha) .$$

3. $\int ((\boldsymbol{v}^\alpha)') \circ \tilde{\boldsymbol{w}}^\alpha$ represents the deformation of the vorticity field $\tilde{\boldsymbol{w}}^\alpha$ generated by all vortons except $\alpha$, due to the velocity field $\boldsymbol{v}^\alpha$ generated by vorton $\alpha$ at the location of vorton $\alpha$.

   Applying both (C.2) and (C.3), and rewriting this integral as

   $$((\boldsymbol{v}^\alpha)') \circ \int \tilde{\boldsymbol{w}}^\alpha ,$$

   we get:

   $$-\sum_{\beta \neq \alpha} \boldsymbol{\gamma}_\alpha \times [(\phi''(\boldsymbol{R}_{\alpha\beta}) \circ (-\frac{1}{3}\boldsymbol{\gamma}_\beta)] .$$

[3] Here $\phi$ is the function defined in (8.9) and definitions (8.12) and (8.13) have been applied.

4. $\int ((\tilde{\boldsymbol{v}}^\alpha)') \circ \tilde{\boldsymbol{w}}^\alpha$ represents the deformation of the vorticity field $\tilde{\boldsymbol{w}}^\alpha$ due to the velocity field $\tilde{\boldsymbol{v}}^\alpha$ at the location of vorton $\alpha$.

   Both fields $\tilde{\boldsymbol{v}}^\alpha$ and $\tilde{\boldsymbol{w}}^\alpha$ are continuous at location $\boldsymbol{r}_\alpha$. Therefore, by the mean value theorem, integration leads to an expression of order $\epsilon^3$, where $\epsilon$ is the radius of sphere $\beta_\alpha$. Hence, it can be disregarded since $\epsilon \ll 1$.

Taking together all the contributing parts, we get the vorton deformation equation:

$$\dot{\boldsymbol{\gamma}}_\alpha = \sum_{\beta \neq \alpha} \{-\boldsymbol{\gamma}_\beta \times (\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\alpha) + \frac{1}{2}\boldsymbol{\gamma}_\alpha \times (\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\beta)\}$$

or, in full:

$$\dot{\boldsymbol{\gamma}}_\alpha = \frac{3}{4\pi} \sum_{\beta \neq \alpha} \{\frac{1}{2}\frac{\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha}{R^3_{\alpha\beta}} + \frac{(\boldsymbol{R}_{\alpha\beta} \times \boldsymbol{\gamma}_\beta)(\boldsymbol{\gamma}_\alpha \cdot \boldsymbol{R}_{\alpha\beta})}{R^5_{\alpha\beta}} + \frac{1}{2}\frac{(\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_{\alpha\beta})(\boldsymbol{\gamma}_\beta \cdot \boldsymbol{R}_{\alpha\beta})}{R^5_{\alpha\beta}}\} . \tag{C.4}$$

In the same way, starting from the transposed Helmholtz equation (8.3) and making use of a rule similar to (C.3), i.e.:

$$((\boldsymbol{a} \times \boldsymbol{\nabla}\phi(\boldsymbol{x}))')^* \circ \boldsymbol{b} = \phi''(\boldsymbol{x}) \circ (\boldsymbol{b} \times \boldsymbol{a}) ,$$

we derive:

$$\dot{\boldsymbol{\gamma}}_\alpha = \sum_{\beta \neq \alpha} \{(\phi''(\boldsymbol{R}_{\alpha\beta}) \circ (\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha) - \frac{1}{2}\boldsymbol{\gamma}_\alpha \times ((\phi''(\boldsymbol{R}_{\alpha\beta}) \circ \boldsymbol{\gamma}_\beta)\} \tag{C.5}$$

or, in full:

$$\dot{\boldsymbol{\gamma}}_\alpha = \frac{3}{4\pi} \sum_{\beta \neq \alpha} \{-\frac{1}{2}\frac{\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha}{R^3_{\alpha\beta}} + \frac{\boldsymbol{R}_{\alpha\beta}[\boldsymbol{R}_{\alpha\beta} \cdot (\boldsymbol{\gamma}_\beta \times \boldsymbol{\gamma}_\alpha)]}{R^5_{\alpha\beta}} - \frac{1}{2}\frac{(\boldsymbol{\gamma}_\alpha \times \boldsymbol{R}_{\alpha\beta})(\boldsymbol{\gamma}_\beta \cdot \boldsymbol{R}_{\alpha\beta})}{R^5_{\alpha\beta}}\} . \tag{C.6}$$

# Symbols

| | | |
|---|---|---|
| $a$ | $=$ | vortex ring core radius (fig.2.3) |
| $\tilde{a}$ | $=$ | non-dimensionalized vortex ring core radius (10.1) |
| $\boldsymbol{A}$ | $=$ | vector potential (8.8) |
| $B_\alpha$ | $=$ | volume of radius $\epsilon$ around vorton location $\boldsymbol{r}_\alpha$ (8.16) |
| $E$ | $=$ | total kinetic energy (9.9) |
| $E(k)$ | $=$ | energy spectrum (9.11) |
| $E_i$ | $=$ | interaction energy (9.10) |
| $E_i(k)$ | $=$ | interaction energy spectrum (9.13) |
| $E_0$ | $=$ | self-energy (9.14) |
| $E_0(k)$ | $=$ | self-energy spectrum (9.12) |
| $H$ | $=$ | total helicity (9.15) |
| $H_i$ | $=$ | interaction helicity (9.16) |
| $\boldsymbol{J}$ | $=$ | total angular momentum (9.6) |
| $k$ | $=$ | wave-number |
| $\boldsymbol{n}$ | $=$ | outward normal unit vector |
| $N$ | $=$ | number of vortons in a vorton ring |
| $\boldsymbol{P}$ | $=$ | total linear momentum (9.2) |
| $\boldsymbol{r}_\alpha$ | $=$ | location vector of a vorton labelled $\alpha$ |
| $R$ | $=$ | vortex ring radius (fig.2.3) |
| $\boldsymbol{R}_\alpha$ | $\equiv$ | $\boldsymbol{x} - \boldsymbol{r}_\alpha$ |
| $\boldsymbol{R}_{\alpha\beta}$ | $\equiv$ | $\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta$ |
| Re | $=$ | Reynolds number (§A.3 of Interlude) |
| $t$ | $=$ | time |
| $u(y)$ | $=$ | shear flow velocity profile (10.8) |
| $U$ | $=$ | outer shear flow velocity (10.8) |
| $\boldsymbol{v}$ | $=$ | velocity |
| $V$ | $=$ | vortex ring velocity (fig.2.3) |
| $\tilde{V}$ | $=$ | non-dimensionalized vortex ring velocity (10.5) |
| $\boldsymbol{w}$ | $=$ | vorticity (1.1) |
| $\bar{\boldsymbol{w}}$ | $=$ | diagnostic vorticity (9.18) |
| $\boldsymbol{x}$ | $=$ | spatial location |
| $x, y, z$ | $=$ | components of $\boldsymbol{x}$ |
| $\boldsymbol{X}$ | $=$ | material location |
| $X_1, X_2, X_3$ | $=$ | components of $\boldsymbol{X}$ |
| $\alpha, \beta$ | : | labels of vortons |
| $\boldsymbol{\gamma}_\alpha$ | $=$ | strength vector of a vorton labelled $\alpha$ |
| $\Gamma$ | $=$ | circulation (4.2) |
| $\delta$ | $=$ | shear flow height (10.8) |
| $\phi(\boldsymbol{x})$ | $\equiv$ | $1/(4\pi x)$ (8.9) |
| $\boldsymbol{\omega}$ | $=$ | angular velocity (2.1) |

| | | |
|---|---|---|
| $\boldsymbol{\nabla}$ | = | spatial nabla operator (1.1) |
| $\frac{\partial}{\partial t}$ | = | spatial derivative |
| $D\ /Dt$ | = | material derivative (1.3) |
| $\delta(...)$ | = | Dirac delta function |
| $\cdot$ | = | scalar product |
| $\times$ | = | vector product |
| $(\boldsymbol{v}')$ | : | deformation matrix (8.2) |
| $(\boldsymbol{v}')^*$ | : | transposed of matrix $(\boldsymbol{v}')$ |
| $\int_V$ | : | volume integral |
| $\int_{\partial V}$ | : | surface integral |
| $\oint_C$ | : | contour integral |

# Bibliography

- *MPP* = W. Thomson, *Mathematical and Physical Papers.* Vols. I-VI. Cambridge, 1882-1911.
- *SPM* = W.D. Niven (ed.), *The Scientific Papers of James Clerk Maxwell.* Vols. I-II. Camdridge, 1890 (reprint: 1965).
- *PMPS* = O. Reynolds, *Papers on Mechanical and Physical Subjects.* Vols. I-III. Cambridge, 1900-1901.
- *MPPL* = J. Larmor, *Mathematical and Physical Papers.* Vols. I-II. Cambridge, 1929.
- *BAAS Rep.* = *Annual Report* of the British Association for the Advancement of Science.
- *JFM* = *Journal of Fluid Mechanics*
- *PhF* = *Physics of Fluids (A)*


[1] R. Aarts, *A numerical study of quantized vortices in He II.* Thesis. Eindhoven, 1993.

[2] M.S. Acarlar, C.R. Smith, "A study of hairpin vortices in a laminar boundary layer. Part 1. Hairpin vortices generated by a hemisphere protuberance". *JFM* **175** (1987) 1-41.

[3] D.J. Acheson, *Elementary fluid dynamics.* Oxford, 1990.

[4] E.J. Aiton, *The vortex theory of planetary motions.* New York, 1972.

[5] M.I. Aksman, E.A. Novikov, "Reconnections of vortex filaments". *Fluid Dyn. Res.* **3** (1988) 239.

[6] M.J. Aksman, E.A. Novikov, S.A. Orszag, "Vorton method in three-dimensional hydrodynamics". *Phys. Rev. Lett.* **55** (1985) 2510.

[7] A. Ambrosetti, M. Struwe, "Existence of steady vortex rings in an ideal fluid". *Arch. Rat. Mech. Anal.* **108** (1989) 97-109.

[8] C. Anderson, C. Greengard, "On vortex methods". *SIAM J. Numer. Anal.* **22** (1985) 413.

[9] C. Anderson, C. Greengard (eds.), *Vortex methods.* Berlin, etc., 1988.

[10] C.R. Anderson, C. Greengard (eds.), *Vortex Dynamics and Vortex Methods.* Providence, 1991.

[11] H. Aref, "Integrable chaotic and turbulent vortex motion in two-dimensional flows". *Ann. Rev. Fluid Mech.* **15** (1983) 345-390.

[12] H. Aref, E.P. Flinchem, "Dynamics of a vortex filament in a shear flow". *JFM* **148** (1984) 477-497.

[13] H. Aref, T. Kambe, "Report on the IUTAM symposium: fundamental aspects of vortex motion". *JFM* **190** (1988) 571-595.

[14] H. Aref, N. Rott, H. Thomann, "Gröbli's solution of the three-vortex problem". *Ann. Rev. of Fluid Mech.* **24** (1992) 1-20.

[15] H. Aref, I. Zawadzki, "Linking of vortex rings". *Nature* **354** (1991) 50-53.

[16] Th. Dracos, A. Tsinober (eds.), *New Approaches and Concepts in Turbulence.* Basel, etc., 1993.

[17] O.M. Ashford, *Prophet or Professor? The Life and Work of Lewis Fry Richardson.* Bristol, etc., 1985.

[18] W.T. Ashurst, D.I. Meiron, "Numerical study of vortex reconnection". *Phys. Rev. Lett.* **58** (1987) 1632.

[19] R.S. Ball, "Account of experiments upon the retardation experienced by vortex rings of air when moving through air". *Trans. Roy. Irish Acad.* **25** (1872) 135-155.

[20] G.K. Batchelor, "Geoffrey Ingram Taylor". *JFM* **173** (1986) 1-14.

[21] J.T. Beale, "A convergent 3-D vortex method with grid-free stretching". *Math. Comp.* **46** (1986) 401-424.

[22] J.T. Beale, A. Majda, "Vortex methods I: convergence in three dimensions". *Math. Comp.* **39** (1982) 1-27.

[23] R. Benzi, S. Patarnello, P. Santangelo, "On the statistical properties of two-dimensional decaying turbulence". *Europhys. Lett.* **3** (1987) 811-818.

[24] M.A. Berger, G.B. Field, "The topological properties of magnetic helicity". *JFM* **147** (1984) 133-148.

[25] J. Bertrand, "Renaissance de la physique Cartésienne". *J. des Savants* (1870) 445-464.

[26] D. Bobylew, "Einige Betrachtungen über die Gleichungen der Hydrodynamik". *Math. Ann.* **6** (1873) 72-84.

[27] O.N. Boratav, R.B. Pelz, N.J. Zabusky, "Reconnection in orthogonally interacting vortex tubes: direct numerical simulations and quantifications". *PhF* **4** (1992) 581.

[28] A.J. Brand, *The turbulent boundary layer.* Thesis. Delft, 1992.

[29] B.C. Brodie, "Discussion on Dr. Williamson's lecture on the atomic theory". *J. of the Chem. Soc. (London)* **22** (1869) 433-441.

[30] S.G. Brush, "Theories of liquid viscosity". *Chem. Rev.* **62** (1962) 513-543.

[31] S.G. Brush, *The kind of motion we call heat.* Amsterdam, etc., 1976.

[32] R.E. Caflisch (ed.), *Mathematical Aspects of Vortex Dynamics.* Philadelphia, 1989.

[33] R.E. Caflisch, G.C. Papanicolaou (eds.), *Singularities in Fluids, Plasmas and Optics.* Dordrecht, etc., 1993.

[34] D.S. Chapman, P.R. Critchlow, "Formation of vortex rings from falling drops". *JFM* **29** (1967) 177-185.

[35] J.A.V. Chapple, *Science and literature in the nineteenth century.* Basingstoke, 1986.

[36] S.G. Chefranov, "Dynamics of point vortex dipoles and spontaneous singularities in three-dimensional turbulent flows". *Sov. Phys. JETP* **66** (1987) 85-89.

[37] A.J. Chorin, "Vortex models and boundary layer instability". *SIAM J. Sci. Stat. Comput.* **1** (1980) 1-21.

[38] C. Chree, "Vortices in a compressible fluid". *Mess. Math.* **17** (1888) 105.

[39] C.C. Chu, R.E. Flaco, "Vortex ring/viscous wall layer interaction model of the turbulence production process near walls". *Exp. Fl.* **6** (1988) 305-315.

[40] C. Chu, C. Wang, C. Hsieh, "An experimental investigation of vortex motion near surfaces". *PhF* **5** (1993) 662-676.

[41] G.-H. Cottet, "A new approach for the analysis of Vortex Methods in two and three dimensions". *Ann. Inst. Henri Poincaré* **5** (1988) 227-285.

[42] G.H. Cottet, J. Goodman, T.Y. Hou, "Convergence of the grid-free point vortex method for the 3-D Euler equations". *SIAM J. Numer. Anal* **28** (1991) 291-307.

[43] J. Croll, "Note on the vortex-atom theory". *Amer. J. Sci.* **26** (1883) 478-9.

[44] D.B. van Dam, *Een test van de vortonmethode door de simulatie van en experimenten met pseudo-ellipsvormige wervelringen.* Report Lab. for Aero- and Hydrodynamics MEAH-115. Delft, 1993.

[45] R.L. Davis, E.P.S. Shellard, "Cosmic vortons". *Nucl. Phys. B* **323** (1989) 209-224.

[46] M.R. Dhanak, "Stability of a regular polygon of finite vortices". *JFM* **234** (1992) 297-316.

[47] E.J. Dijksterhuis, *De mechanisering van het wereldbeeld.* Amsterdam, 1985[5] (English edition: London, etc., 1961).

[48] R.J. Donnelly, "Superfluid turbulence". *Sci. Am.* **259**(5) (1988) 77.

[49] D.G. Dritschel, "The stability and energetics of corotating uniform vortices". *JFM* **157** (1985) 95-134.

[50] R. Dugas, *Histoire de la Mécanique.* Neuchâtel, 1950.

[51] P. Duhem, "L'école anglaise et les théories physiques". *Rev. des Quest. Sci.* **34** (1893) 345-378.

[52] P. Duhem, *L'evolution de la mécanique.* Paris, 1905 (English edition: Alphen aan den Rijn, etc., 1980).

[53] F. Durst, B.E. Launder, J.L. Lumley, F.W. Schmidt, J.H. Whitelaw (eds.), *Turbulent Shear Flows 5.* Berlin, etc., 1987.

[54] F.W. Dyson, "The potential of an anchor ring - Part II". *Phil. Trans. A* **184** (1893) 1041-1106.

[55] A. Eliassen, "Vilhelm Bjerknes and his students". *Ann. Rev. Fluid Mech.* **14** (1982) 1-11.

[56] G.F. FitzGerald, "On currents of gas in the vortex atom theory of gases". *Dublin Soc. Sci. Proc.* **4** (1885) 339-340.

[57] G.F. FitzGerald, "On a model illustrating properties of ether". *Dublin Soc. Sci. Proc.* **4** (1885) 407-419.

[58] G.F. FitzGerald, "Address". *BAAS Rep.* (1888) 557-563.

[59] G.F. FitzGerald, "On an electro-magnetic interpretation of turbulent liquid motion". *Nature* **40** (1889) 32-34.

[60] W. Frost, T.H. Moulden (eds.), *Handbook of Turbulence.* Vol. I. New York, etc., 1977.

[61] D. Fryberger, "A semiclassical monopole configuration for electromagnetism". *Hadr. J.* **4** (1981) 1844-1888.

[62] M. Gad-el-Hak, A.K.M.F. Hussain, "Coherent structures in a turbulent boundary layer. Part 1: Generation of artificial bursts". *PhF* **29** (1986) 2124.

[63] A. Gautier, "Les problèmes de la chimie moderne". *Rev. Gen. des Sci. P. et A.* **1** (1890) 229.

[64] R. Giacomelli, E. Pistolesi, "Historical sketch". In: *Aerodynamic Theory* I. Berlin, 1934.

[65] C.C. Gillispie (ed.), *Dictionary of Scientific Biography.* New York, 1980.

[66] M. Goldman, *The demon in the aether; the story of James Clerk Maxwell.* Edinburgh, 1983.

[67] S. Goldstein, "Fluid mechanics in the first half of this century". *Ann. Rev. Fluid Mech.* **1** (1969) 1-28.

[68] A.J. Grass, R.J. Stuart, M. Mansour-Tehrani, "Vortical structures and coherent motion in turbulernt flow over smooth and rough boundaries". *Phil. Trans. R. Soc. London A* **336** (1991) 35-65.

[69] A.G. Gray, "Determination of the translational velocity of a vortex ring of small cross-section". *Phil. Mag.* **28** (1914) 13-18.

[70] C. A. Greengard, *Three-Dimensional Vortex Methods.* Thesis. Berkeley, 1984.

[71] C. Greengard, E. Thomann, "Singular vortex systems and weak solutions of the Euler equations". *PhF* **31** (1988) 2810.

[72] E. van Groesen, E.M. de Jager, *Mathematical Structures in Continuous Dynamical Systems.* Amsterdam, 1994.

[73] A. Gyr (ed.), *Structure of Turbulence and Drag Reduction.* Berlin, etc., 1990.

[74] M.R. Head, P. Bandyopadhyay, "New aspects of turbulent boundary layer structure". *JFM* **107** (1981) 297.

[75] H. Helmholtz, "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen". *J. Reine Angew. Math.* **55** (1858) 25-55 (English translation: *Phil. Mag.* **33** (1867) 485-512).

[76] H. Helmholtz, *Populäre wissenschaftliche Vorträge.* Braunschweig, 1876.

[77] H. Hertz, *Die Prinzipien der Mechanik. Gesammelte Werke von Heinrich Hertz.* Leipzig, 1894.

[78] W.M. Hicks, "On the problem of two pulsating spheres in a fluid". *Proc. Camb. Phil. Soc.* **3** (1879) 276-285.

[79] W.M. Hicks, "Report on recent progress in hydrodynamics - Part I". *BAAS Rep.* (1880) 57-88.

[80] W.M. Hicks, "On the steady motion of a hollow vortex". *Proc. R. Soc. London* **35** (1883) 304-308.

[81] W.M. Hicks, "On the steady motion and small vibrations of a hollow vortex". *Phil. Trans. R. Soc. London* **175** (1884) 161-195.

[82] W.M. Hicks, "Researches on the theory of vortex rings - Part II". *Phil. Trans. R. Soc. London* **176** (1885) 725-780.

[83] W.M. Hicks, "On the constitution of the luminiferous ether on the vortex atom". *BAAS Rep.* (1885) 930.

[84] W.M. Hicks, "A vortex analogue of static electricity". *BAAS Rep.* (1888) 577.

[85] W.M. Hicks, "Address". *BAAS Rep.* (1895) 595-606.

[86] W.M. Hicks, "Researches in vortex motion". *Proc. R. Soc. London* **62** (1898) 332-338.

[87] W.M. Hicks, "Researches in vortex motion - Part III. On spiral or gyrostatic vortex aggregates". *Phil. Trans. R. Soc. London A* **192** (1899) 33-101.

[88] W.M. Hicks, "On the mutual threading of vortex rings". *Proc. R. Soc. London A* **102** (1922) 111-131.

[89] J.O. Hinze, *Turbulence.* New York, etc., 1975².

[90] T.Y. Hou, J. Lowengrub, "Convergence of the point vortex method for the 3-D Euler equations". *Comm. Pure Appl. Math.* **43** (1990) 965-981.

[91] L. Howarth, "One hundred years of Lamb's Hydrodynamics". *JFM* **90** (1979) 202-207.

[92] A.K.M.F. Hussain, "Coherent structures - reality and myth". *PhF* **26** (1983) 2816.

[93] N. Izutsu, Y. Oshima, *Hot Wire Measurement of Cross-linking and Unlinking in Uniting Two Circular Vortex Rings. Inst. of Space & Astronaut. Sci. Report* No. 638, 1991.

[94] F. Jenkin, "Lucretius and the atomic theory". In: *Papers literary, scientific,* vol. I. London, 1887.

[95] J. Jiménez, A.A. Wray, P.G. Saffman, R.S. Rogallo, "The structure of intense vorticity in isotropic turbulence". *JFM* **255** (1993) 65-90.

[96] V.A. Julius, "Sur les spectres des lignes des éléments". *Ann. de l'Ecole Polytechnique* **5** (1889) 1-117.

[97] W.H. Julius, *Het warmtespectrum en de trillingsperioden der moleculen van eenige gassen.* Thesis. Utrecht, 1888.

[98] T. Kambe, U Mya Oo, "An axisymmetric viscous vortex motion and its acoustic emission". *J. Phys. Soc. Japan* **53** (1984) 2263-2273.

[99] R. Kargon, P. Achinstein (eds.), *Kelvin's Baltimore Lectures and Modern Theoretical Physics; Historical and Philosophical Perspectives.* Cambridge (MA), etc., 1987.

[100] Th. von Kármán, L. Edson, *The Wind and Beyond.* Boston, etc., 1967.

[101] M.J. Kascic, "Vorton dynamics: a case study of developing a fluid dynamics model for a vector processor". *Parallel Computing* **1** (1985) 35.

[102] S. Kida, M. Takaoka, "Reconnection of vortex tubes". *Fluid Dyn. Res.* **3** (1988) 257-261.

[103] S. Kida, M. Takaoka, F. Hussain, "Collision of two vortex rings". *JFM* **230** (1991) 583-646.

[104] P.W. Kingsford, *F.W. Lanchester.* London, 1960.

[105] M. Kiya, H. Ishii, "Vortex interaction and Kolmogorov spectrum". *Fluid Dyn. Res.* **8** (1991) 73-83.

[106] P.S. Klebanoff, W.G. Cleveland, K.D. Tidstron, "On the evolution of a turbulent boundary layer induced by a three-dimensional roughness element". *JFM* **237** (1992) 101-187.

[107] S.J. Kline, N.H. Afgan (eds.), *Near-Wall Turbulence.* New York, etc., 1988.

[108] O.M. Knio, A.F. Ghoniem, "Numerical study of a three-dimensional vortex method". *J. Comp. Phys.* **86** (1990) 75-106.

[109] C.G. Knott, *Life and scientific work of Peter Guthrie Tait.* Cambridge, 1911.

[110] L. Koenigsberger, *Hermann von Helmholtz*. Braunschweig, 1902-03.

[111] L. Koenigsberger, *H. von Helmholtz's Untersuchungen über die Grundlagen der Mathematik und Mechanik*. Leipzig, 1896 (English edition: New York, 1965).

[112] C.-H. Krutzsch, "Über eine experimentell beobachtete Erscheinung an Wirbelringen bei ihrer translatorischen Bewegung in wirklichen Flüssigkeiten". *Ann. Phys.* **35** (1939) 497-523.

[113] S. Kuwabara, "Studies on vorton motion based on the dividing vorton model". *Inst. of Space & Astronaut. Sci. Report* SP4, 1986.

[114] S. Kuwabara, "Pseudo-canonical formulation of 3-dimensional vortex motion and vorton model analysis". *Fluid Dyn. Res.* **3** (1988) 163.

[115] H. Lamb, *Hydrodynamics* (original title: *A Treatise on the Mathematical Theory of the Motion of Fluids*). Cambridge, 1879, $1895^2$, $1906^3$, $1916^4$.

[116] J. Larmor, "A dynamical theory of the electric and luminiferous medium - Part I" (1893). *MPPL*, vol. I, §§37 & 38.

[117] J. Larmor, "Address". *BAAS Rep.* (1900) 613-628.

[118] J. Larmor, *Aether and matter*. Cambridge, 1900.

[119] J. Larmor, "The methods of mathematical physics" (1900). *MPPL*, vol. II, §54.

[120] K. Lasswitz, "Über Wirbelatome und stetige Raumerfüllung". *Vierteljahrschr. f. wissenschaftl. Philosophie* **3** (1879) 206-215 & 275-294.

[121] A. Leonard, "Computing three-dimensional incompressible flows with vortex elements". *Ann. Rev. Fluid Mech.* **17** (1985) 523.

[122] A. Leonard, "Vortex Methods for flow simulation". *J. Comp. Phys.* **37** (1980) 289-335.

[123] A. Leonard, K. Chua, "Three-dimensional interaction of vortex tubes". *Physica D* **37** (1989) 490-496.

[124] E. Levich, "Certain problems in the theory of developed hydrodynamical turbulence". *Physics Reports* Nos. 3 & 4 (1987) 129-238.

[125] M. Lévy, "L'hydrodynamique moderne et l'hypothèse des actions à distance". *Rev. Gen. des Sc. Pures et Appl.* **1** (1890) 721.

[126] A. Levy, B. Forsdyke, "The steady motion and stability of a helical vortex". *Proc. R. Soc. London A* **120** (1928) 670-690.

[127] T.T. Lim, "An experimental study of a vortex ring interacting with an inclined wall". *Exp. Fl.* **7** (1989) 453-463.

[128] T.T. Lim, T.B. Nickels, "Instability and reconnection in the head-on collision of two vortex rings". *Nature* **357** (1992) 225-226.

[129] T.T. Lim, T.B. Nickels, M.S. Chong, "A note on the cause of rebound in the head-on collision of a vortex ring with a wall". *Exp. in Fluids* **12** (1991) 41-48.

[130] O. Lodge, "The stream-lines of moving vortex-rings". *Phil. Mag.* **20** (1885) 67-70.

[131] O. Lodge, "Aberration problems". *Phil. Trans. R. Soc.* **184** (1893) 727-804.

[132] H.A. Lorentz, "De moleculaire theoriën in de natuurkunde". *Collected Papers* (The Hague, 1939), vol. IX, 26-49.

[133] A.E.H. Love, "On recent English researches in vortex-motion". *Math. Ann.* **30** (1887) 326-344.

[134] H.J. Lugt, *Vortex flow in nature and technology*. New York, 1983.

[135] J.L. Lumley (ed.), *Whither Turbulence? Turbulence at the Crossroads*. Berlin, etc., 1990.

[136] T.S. Lundgren, "Strained spiral vortex mode for turbulent fine structure". *PhF* **25** (1982) 2193-2203.

[137] F. van Lunteren, *Framing hypotheses; Conceptions of gravity in the 18th and 19th centuries*. Thesis. Utrecht, 1991.

[138] A.J. Majda, "Vorticity, turbulence, and acoustics in fluid flow". *SIAM Review* **33** (1991) 349-388.

[139] C. Marchioro, M. Pulvirenti, *Vortex Methods in Two-dimensional Fluid Dynamics.* Berlin, 1984.

[140] C. Marchioro, M. Pulvirenti, *Mathematical Theory of Incompressible Non-Viscous Fluids.* (to be published)

[141] C. Marchioro, M. Pulvirenti, "Vortices and localization in Euler flows". Internal report. Rome, 1992.

[142] J. Clerk Maxwell, "On Faraday's lines of force" (1856). *SPM*, vol. I, §VIII.

[143] J. Clerk Maxwell, "On physical lines of force" (1861). *SPM*, vol. I, §XXIII.

[144] J. Clerk Maxwell, "Review". *Nature* **16** (1877) 242-246.

[145] J. Clerk Maxwell, "Address" (1870). *SPM*, vol. II, §XLI.

[146] J. Clerk Maxwell, "Molecules" (1873). *SPM*, vol. II, §LXII.

[147] J. Clerk Maxwell, "Atom" (1874). *SPM*, vol. II, §LXXIII.

[148] T. Maxworthy, "The structure and stability of vortex rings". *JFM* **51** (1972) 15-32.

[149] T. Maxworthy, "Some experimental studies of vortex rings". *JFM* **81** (1977) 465-495.

[150] D.M. McDowell, J.D. Jackson, *Osborne Reynolds and Engineering Science Today.* Manchester, 1970.

[151] W. McGucken, *Nineteenth-Century Spectroscopy; Development of the Understanding of Spectra 1802-1897.* Baltimore, etc., 1969.

[152] M.V. Melander, F. Hussain, "Cross-linking of two anti-parallel vortex tubes". *PhF* **1** (1989) 633-636.

[153] V.V. Meleshko, M.J. Konstantinov, *Dynamics of vortex structures.* Kiev, 1993. (in Russian)

[154] D. Mendeleeff, *The Principles of Chemistry.* London, 1891.

[155] O.E. Meyer, *Die kinetische Theorie der Gase.* Breslau, 1877, $1899^2$.

[156] A.A. Michelson, *Light waves and their uses.* Chicago, 1907.

[157] H.K. Moffatt, "The degree of knottedness of tangled vortex lines". *JFM* **35** (1969) 117-129.

[158] H.K. Moffatt, "On the existence of localized rotational disturbances which propagate without change of structure in an inviscid fluid". *JFM* **173** (1986) 289-302.

[159] H.K. Moffatt, "Structure and stability of solutions of the Euler equations: a lagrangian approach". *Phil. Trans. R. Soc. London A* **333** (1990) 321-342.

[160] H.K. Moffatt, A. Tsinober (eds.), *Topological Fluid Mechanics.* Cambridge, 1990.

[161] H.K. Moffatt, A. Tsinober, "Helicity in laminar and turbulent flow". *Ann. Rev. Fluid Mech.* **24** (1992) 281-312.

[162] H.K. Moffatt, G.M. Zaslavsky, P. Comte, M. Tabor (eds.), *Topological Aspects of the Dynamics of Fluids and Plasmas.* Dordrecht, etc., 1992

[163] P. Moin, A. Leonard, J. Kim. "Evolution of a curved vortex filament into a vortex ring". *PhF* **29** (1986) 955-963.

[164] J.F. Morrison, C.S. Subramanian, P. Bradshaw, "Bursts and the law of the wall in turbulent boundary layers". *JFM* **241** (1992) 75-108.

[165] M.C. Mosher, "A method for computing three-dimensional vortex flows". *Zeitschrift Flugwiss. Weltraumf.* **9** (1985) 125-133.

[166] E.F. Northrup, "An experimental study of vortex motions in liquids". *J. Franklin Inst.*, **172** (1911) 211-226 & 345-368.

[167] E.F. Northrup, "An experimental study of vortex motions in liquids". *Nature* **88** (1912) 463-468.

[168] E.A. Novikov, "Generalized dynamics of three-dimensional vortical singularities (vortons)". *Sov. Phys. JETP* **57** (1983) 566.

[169] E.A. Novikov, "Three-dimensional singular vortical flows in the presence of a boundary". *Phys. Letters A* **112** (1985) 327.

[170] E.A. Novikov, "Transformation of a vortex ring, initiated by a downburst, into a horseshoe vortex in the boundary layer". *Boundary-Layer Meteorology* **38** (1987) 305.

[171] Y. Oshima, N. Izutsu, "Cross-linking of two vortex rings". *PhF* **31** (1988) 2401-2403.

[172] H. Parenty, *Les tourbillons de Descartes et la science moderne.* Paris, 1903.

[173] P.J. Pauly, *Vortices and vibrations: the rise and fall of a scientific research program.* Thesis. Maryland, 1975.

[174] K. Pearson, "On a certain atomic hypothesis". *Trans. Cambr. Phil. Soc.* **14** (1889) 71-120.

[175] K. Pearson, "On a certain atomic hypothesis - ether squirts". *Amer. J. Math.* **13** (1892) 310.

[176] K. Pearson, *The grammar of science.* London, $1900^2$.

[177] G. Pedrizzetti, "Insight into singular vortex flows". *Fluid Dyn. Res.* **10** (1992) 101-116.

[178] G. Pedrizzetti, *Dinamica di fenomeni vorticosi.* Thesis. Firenze, 1992.

[179] H.C. Pocklingtion, "The complete system of the periods of a hollow vortex ring". *Phil. Trans. R. Soc. London* **186** (1895) 603-619

[180] H. Poincaré, *Théorie des Tourbillons.* Paris, 1893.

[181] S.T. Preston, "On the physical aspects of the vortex-atom theory". *Nature* **22** (1880) 56-59.

[182] S.T. Preston, "Vortex Atoms". *Nature* **22** (1880) 121-122.

[183] S.T. Preston, "A psychological aspect of the vortex-atom theory". *Nature* **21** (1880) 323.

[184] N. Quint, *De wervelbeweging.* Thesis. Amsterdam, 1888.

[185] W.J.M. Rankine, "On the Use of Mechanical Hypotheses in Science, and Especially in the Theory of Heat". *Proc. Roy. Phil. Soc. Glasgow* **5** (1864) 126-132.

[186] Lord Rayleigh, *The life of Sir J.J. Thomson.* Cambridge, 1943.

[187] E. Reusch, "Ueber Ringbildung in Flüssigkeiten". *Ann. Physik u. Chemie* **110** (1860) 309-316.

[188] O. Reynolds, "On the resistance encountered by vortex rings, and the relation between the vortex rings and the streamlines of a disc". *Nature* **14** (1876) 477-479.

[189] O. Reynolds, "On vortex motion" (1877). *RP*, vol. I, §25.

[190] O. Reynolds, "Vortex rings". *Nature* **27** (1883) 193-195.

[191] O. Reynolds, "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels" (1883). *RP*, vol. II, §44.

[192] R.L. Ricca, "Rediscovery of Da Rios equations". *Nature* **352** (1991) 561-562.

[193] P.H. Roberts, "A Hamiltonian theory for weakly interacting vortices". *Mathematica* **19** (1972) 169-179.

[194] A.C. Robinson, P.G. Saffman, "Three-dimensional stability of vortex arrays". *JFM* **125** (1982) 411-427.

[195] S.K. Robinson, "Coherent motions in the turbulent boundary layer". *Ann. Rev. Fluid Mech.* **23** (1991) 601-639.

[196] S.K. Robinson, *The Kinematics of Turbulent Boundary Layer Structure.* NASA Technical Memorandum 103859. Moffett Field, 1991.

[197] W.B. Rogers, "On the formation of rotating rings by air and liquids under certain conditions". *Amer. J. Sci.* **26** (1858) 246-258.

[198] M.M. Rogers, P. Moin, "Structure of the vorticity field in homogeneous turbulent flows". *JFM* **176** (1987) 33-66.

[199] L. Rosenhead, "The formation of vortices from a surface of discontinuity". *Proc. R. Soc. A* **134** (1931) 170-191.

[200] N. Rott, "Note on the history of the Reynolds number". *Ann. Rev. Fluid Mech.* **22** (1990) 1-11.

[201] N. Rott, B. Cantwell, "Vortex drift I; Dynamic interpretation". *PhF* **5** (1993) 1443-1450.

[202] J.C. Rotta, *Die Aerodynamische Versuchsanstalt in Göttingen, ein Werk Ludwig Prandtls.* Göttingen, 1990.

[203] H. Rouse, S. Ince, *History of hydraulics.* Iowa City, 1957.

[204] P.G. Saffman, "A model of vortex reconnection". *JFM* **212** (1989) 395-402.

[205] P.G. Saffman, *Vortex Dynamics.* Cambridge, 1992.

[206] P.G. Saffman, D.I. Meiron, "Difficulties with three-dimensional weak solutions for inviscid incompressible flow". *PhF* **29** (1986) 2373.

[207] T. Sarpkaya, "Computational methods with vortices". *J. of Fluids Engineering* **111** (1989) 5-52.

[208] P.R. Schatzle, *An experimental study of fusion of vortex rings.* Thesis. Pasadena, 1987.

[209] M. Serres, *La naissance de la physique dans le texte de Lucrèce; fleuves et turbulences.* Paris, 1977.

[210] J. Serrin, "Mathematical principles of classical fluid mechanics". In: *Encyclopedia of Physics/Handbuch der Physik,* vol. VIII/1. Berlin, etc., 1959.

[211] Z.-S. She, E. Jackson, S.A. Orszag, "Structure and dynamics of homogeneous turbulence: models and simulations". *Proc. R. Soc. London A* **434** (1991) 101-124.

[212] K. Shariff, A. Leonard, "Vortex rings". *Ann. Rev. Fluid Mech.* **24** (1992) 235-279.

[213] H.I. Sharlin, *Lord Kelvin, the dynamic Victorian.* University Park, etc., 1979.

[214] E.D. Siggia, A. Pumir, "Vortex dynamics and the existence of solutions to the Navier-Stokes equations". *PhF* **30** (1987) 1606-1626.

[215] R.H. Silliman, "William Thomson: Smoke Rings and Nineteenth-Century Atomism". *ISIS* **54** (1963) 461-474.

[216] L. Sirovich (ed.), *New Perspectives in Turbulence.* New York, etc., 1991.

[217] C. Smith, "Engineering the universe: William Thomson and Fleeming Jenkin on the nature of matter". *Ann. Sci.* **37** (1980) 387-412.

[218] C. Smith, M.N. Wise, *Energy and Empire.* Cambridge, etc., 1989.

[219] H.A.M. Snelders, "A.M. Mayer's experiments with floating magnets and their use in the atomic theories of matter". *Ann. Sci.* **33** (1976) 67-80.

[220] J.B. Stallo, *The Concepts and Theories of Modern Physics.* London, $1900^4$.

[221] B. Stewart, P.G. Tait, *The Unseen Universe; or Physical Speculations on a Future State.* London, $1875^1$.

[222] G.G. Stokes, "On the theorems of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids". *Trans. Cambr. Phil. Soc.* **8** (1847) 287-319.

[223] S. Strizhevskii, *Nikolai Zhukovsky, Founder of Aerodynamics.* Moscow, 1957.

[224] A. Stückelberger, *Antike Atompysik.* München, 1979.

[225] J.L. Synge, C.C. Lin, "On a statistical model of isotropic turbulence". *Trans. R. Soc. Can.* **37** (1943) 45.

[226] I. Szabó, *Geschichte der mechanischen Prinzipien.* Basel, etc., $1987^3$.

[227] P.G. Tait, "On knots" (1876-7). *Scientific Papers* (Cambridge, 1898), vol. I, §XXXIX.

[228] P.G. Tait, *Properties of matter.* Edinburgh, 1885.

[229] P.G. Tait, *Lectures on some recent advances in physical science.* London, 1876, $1885^2$.

[230] I. Tani, "History of boundary-layer theory". *Ann. Rev. Fluid Mech.* **9** (1977) 87-111.

[231] G.I. Taylor, *The scientific papers of sir Geoffrey Ingram Taylor.* Cambridge, 1960.

[232] S.P. Thompson, *The Life of William Thomson, Baron Kelvin of Largs.* London, 1910.

[233] J.J. Thomson, "Vortex motion in a viscous incompressible fluid". *Mess. Math.* **8** (1879) 174.

[234] J.J. Thomson, *A Treatise on the motion of vortex rings.* London, 1883 (reprint: 1968).

[235] J.J. Thomson, "On a theory of the electric discharge in gases". *Phil. Mag.* **15** (1883), 427.

[236] J.J. Thomson, "On the chemical combination of gases". *Phil. Mag.* **18** (1884), 233.

[237] J.J. Thomson, "The vortex ring theory of gases. On the law of the distribution of energy among the molecules". *Proc. R. Soc. London* **39** (1885) 23-36.

[238] J.J. Thomson, *The corpuscular theory of matter.* London, 1907.

[239] J.J. Thomson, *Recollections and reflections.* London, 1936.

[240] J.J. Thomson, H.F. Newall, "On the formation of vortex rings". *Proc. R. Soc. London* **39** (1885) 417.

[241] W. Thomson, *Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.* London, 1884, $1904^2$.

[242] W. Thomson, "Dynamical illustrations of the magnetic and the helicoidal rotary effects of transparent bodies on polarized light" (1856). Appendix F in *Baltimore Lectures*, $1904^2$.

[243] W. Thomson, "On vortex atoms" (1867). *MPP*, vol. IV, §1.

[244] W. Thomson, "The translatory velocity of a circular vortex ring" (1867). *MPP*, vol. IV, §3.

[245] W. Thomson, "On vortex motion" (1869). *MPP*, vol. IV, §2.

[246] W. Thomson, "On the ultramundane corpuscles of Le Sage". *Proc. R. Soc. Edin.* **7** (1872) 577-589 = *Phil. Mag.* **45** (1873), 321-332.

[247] W. Thomson, "On vortex motion" (1872). *MPP*, vol. IV, §2.

[248] W. Thomson, "Vortex statics" (1875-6). *MPP*, vol. IV, §10.

[249] W. Thomson, "Floating magnets [illustrating vortex-systems]" (1878). *MPP*, vol. IV, §12.

[250] W. Thomson, "Vibrations of a columnar vortex" (1880). *MPP*, vol. IV, §15.

[251] W. Thomson, "On maximum and minimum energy in vortex motion" (1880). *MPP*, vol. IV, §16.

[252] W. Thomson, "On the average pressure due to impulse of vortex-rings on a solid" (1881). *MPP*, vol. IV, §18.

[253] W. Thomson, "Elasticity viewed as possibly a mode of motion" (1881). *MPP*, vol. IV, §46

[254] W. Thomson, "Steps towards a kinetic theory of matter". *BAAS Rep.* (1884) 613-622.

[255] W. Thomson, "On the stability of steady and of periodic fluid motion" (1887). *MPP*, vol. IV, §16.

[256] W. Thomson, "On the vortex theory of the luminiferous aether/On the propagation of laminar motion through a turbulently moving inviscid liquid" (1887). *MPP*, vol. IV, §33.

[257] W. Thomson, "On the stability and small oscillation of a perfect liquid full of nearly straight coreless vortices" (1889). *MPP*, vol. IV, §21.

[258] W. Thomson, "On the duties of ether for electricity and magnetism" (1900). *MPP*, vol. IV, §82.

[259] W. Thomson, "Ninenteeth century clouds over the dynamical theory of heat and light" (1901). Appendix B in *Baltimore Lectures*, $1904^2$.

[260] W. Thomson, "Aepinus atomized" (1901). Appendix E in *Baltimore Lectures*, $1904^2$.

[261] W. Thomson, "Deep water ship-waves" (1905). *MPP*, vol. IV, §37.

[262] W. Thomson, "On the motions of ether produced by collisions of atoms or molecules, containing or not containing electrions" (1907). *MPP*, vol. VI, §266.

[263] D.R. Tilley, J. Tilley, *Superfluidity and Superconductivity.* Bristol, etc., 1990.

[264] L. Ting, R. Klein, *Viscous Vortical Flows.* Berlin, etc., 1991.

[265] G.A. Tokaty, *A history and philosophy of fluid mechanics.* Henley-on-Thames, 1971.

[266] D.R. Topper, "'To reason by Means of Images': J.J. Thomson and the mechanical picture of nature". *Ann. Sci.* **37** (1980) 31-57.

[267] C. Truesdell, "Prologue". In: *Leonhardi Euleri Commentationes Mechanicae ad theoriam corporum fluidorum pertinentes, volumen prius. Leonhardi Euleri Opera Omnia, series secunda "Opera Mechanica et Astronomica" volumen duodecimum.* Lausanne, 1954.

[268] C. Truesdell, *The kinematics of vorticity.* Bloomington, 1954.

[269] C. Truesdell, *Essays in the History of Mechanics.* Berlin, 1968.

[270] F.M. Turner, "Lucretius among the Victorians". *Vict. Stud.* **16** (1972) 329-348.

[271] J. Varin d'Ainvelle, *L'origine tourbillonnaire de l'atome et ses consequences.* Paris, 1924.

[272] G.H. Vatistas, "A note on liquid vortex sloshing and Kelvin's equilibria". *JFM* **217** (1990) 241-248.

[273] P. Orlandi, R. Verzicco, "Vortex rings impinging on walls: axisymmetric and three-dimensional simulations". *JFM* **256** (1993) 615-646.

[274] J.D.A. Walker, C.R. Smith, A.W. Cerra, T.L. Doligalski, "The impact of a vortex ring on a wall". *JFM* **181** (1987) 99-140.

[275] J.M. Wallace, J.-L. Balint, L. Ong, "An experimental study of helicity density in turbulent flows". *PhF* **4** (1992) 2013.

[276] S.E. Widnall, "The structure and dynamics of vortex filaments". *Ann. Rev. Fluid Mech.* **7** (1975) 141-165.

[277] S.E. Widnall, D.B. Bliss, C.Y. Tsai, "The instability of short waves on a vortex ring". *JFM* **66** (1974) 35-47.

[278] S.E. Widnall, J.P. Sullivan, "On the stability of vortex rings". *Proc. R. Soc. London A* **332** (1973) 335-353.

[279] S.E. Widnall, C. Tsai, "The instability of the thin vortex ring of constant vorticity". *Phil. Trans. R. Soc. London A* **287** (1977) 273-305.

[280] A.N. Williamson, "On the Atomic Theory". *J. of the Chem. Soc. (London)* **22** (1869) 328-365.

[281] D.B. Wilson, *Kelvin and Stokes.* Bristol, 1987.

[282] D.B. Wilson (ed.), *The correspondence between Sir George Gabriel Stokes and Sir William Thomson, Baron Kelvin of Largs.* Vol. I (1846-1869). Cambridge, 1990.

[283] G. S. Winckelmans, *Topics in Vortex Methods for the Computation of Three- and Two-Dimensional Incompressible Unsteady Flows.* Thesis. Pasadena, 1989.

[284] G. Winckelmans, A. Leonard, "Weak solutions of the three-dimensional vorticity equation with vortex singularities". *PhF* **31** (1988) 1838.

[285] R.W. Wood, "Vortex rings". *Nature* **63** (1901) 418-420.

[286] A. Wurtz, *La theorie atomistique.* Paris, 1873 (English edition: London, 1880).

[287] H. Yamada, T. Matsui, "Preliminary study of mutual slip-through of a pair of vortices". *PhF* **21** (1978) 292-294.

[288] N.J. Zabusky, O.N. Boratav, R.B. Pelz, M. Gao, D. Silver, S.P. Cooper, "Emergence of coherent patterns of vortex stretching during reconnection: a scattering paradigma". *Phys. Rev. Lett.* **67** (1991) 2469.

## Samenvatting van *Over Wervelatomen en Vortonen*

Dit proefschrift bestaat uit twee delen. In het eerste wordt de ontwikkeling van een 19e-eeuws atoommodel, het wervelatoom (*vortex atom*), behandeld. In het tweede deel wordt de recent geïntroduceerde vortonmethode besproken en de numerieke simulaties die hiermee zijn uitgevoerd om haar te testen. In een zgn. *Interlude* worden enige ontwikkelingen in de stromingsleer geschetst uit de tussenliggende tijd die belangrijk zijn voor het begrijpen van onderdelen van het tweede deel.

De ontwikkeling van het wervelatoom kan alleen begrepen worden als men enig zicht heeft op de ontwikkelingen op het gebied van materietheorieën en van het begrip vorticiteit binnen de stromingsleer. Op het moment dat het wervelatoom in 1867 door Kelvin werd geïntroduceerd bestond vorticiteit en de bijbehorende theorie nog pas enkele decennia. Hoewel geleerden als Cauchy en Stokes rond 1845 al enige bijdragen hadden geleverd, legde pas in 1858 Helmholtz de basis van de vorticiteitstheorie. Aan hem danken we enige belangrijke definities, theorema's, vergelijkingen en fysische inzichten. Op het gebied van materietheorieën moeten we het klassieke atoom van Democritus noemen. Het atoom als hard, onveranderlijk bolletje was ook in de 19e eeuw nog populair, met name dankzij de opkomst van de kinetische-gastheorie.

De ontwikkeling van Kelvins ideeën op het gebied van de zgn. ether, de hydrodynamica en het electro-magnetisme, samen met de invloed van Faraday, Rankine (het *molecular vortex*) en Stokes, zijn belangrijk voor een goed begrip van de introductie van het wervelatoom door Kelvin. Daarnaast moet als directe aanleiding Taits experiment met rookringen genoemd worden. De elastische wervelringen, zo meende Kelvin, konden veel beter de diverse eigenschappen van materie beschrijven dan het Luctrius atoom: onverwoestbaarheid, zwaartekracht en inertie, spectra.

Kelvins model werd echter over het algemeen koel ontvangen, zeker op het Continent, waar men vooral filosofische bezwaren tegen Kelvins manier van modelvorming had. In Groot-Brittannië probeerden enkele aanhangers langs analytische weg aan te tonen dat het wervelatoom inderdaad belangijke voordelen bood, maar slaagden hierin nauwelijks. Daarbij werden zij vooral gehinderd door gebrek aan wiskundige technieken. Intussen groeide het aantal pogingen aan te tonen dat het model niet bruikbaar was en dat de beweringen van aanhangers obscuur waren. Ook aanpassingen van Kelvins oorspronkelijke atoom leverden niets op.

Pogingen om het wervelatoommodel toe te passen in ethermodellen liepen ook op niets uit en Kelvin verloor het geloof in zijn eigen schepping. Zo kwam hij onder andere tot het inzicht dat wervelringen niet altijd stabiel hoefden te zijn. De ontdekking van het electron en het inzicht dat electrische lading fundamenteel voor een atoommodel was, gaf Kelvins model de genadeslag. Daarnaast was Kelvins manier van modelvorming uit de mode geraakt.

De wervelatoomtheorie, hoe onvruchtbaar verder ook voor de atoomtheorie, heeft een belangrijke stimulans betekend voor het onderzoek naar vorticiteit en wervelstructuren. Zo zijn inmiddels vele, meer geraffineerde, wervelringexperimenten uitgevoerd, de analytische uitwerking is verder ontwikkeld en met name m.b.t. de stabiliteit van wervelringen werden opmerkelijke resultaten gevonden (Kelvins model van de wervelring bleek niet stabiel). Ook werd vorticiteit een vertrouwd begrip in het onderzoek naar turbulente stromingen, waar een van de belangrijkste ontwikkelingen de ontdekking van coherente (wervel)structuren is. Daarnaast hebben in de moderne stromingsleer begrippen als heliciteit en wervelreconnectie zich een belangrijke plaats verworven in de zgn. *topological fluid mechanics.*

Een andere belangrijke moderne ontwikkeling is de opkomst van het gebruik van de computer: "computational fluid mechanics". Daarbinnen valt de opkomst van de zgn. wervelmethoden (*vortex methods*): het numeriek simuleren van wervelstructuren d.m.v. modellering met "wervelelementen" (bijv. *vortex-filaments* en *vortex-points*). We moeten echter wel enige eisen opleggen aan deze methoden: een divergentievrij vorticiteitsveld, correcte modellering van de vorticiteitsdistributie, correcte modellering van de deformatie en interactie van wervelstructuren, behoud van bewegingsinvarianten, geen negatieve gevolgen van *remeshing*, correcte oplegging van randvoorwaarden, convergentie, en aanvaardbare rekeninspanningen. Eén van de recente wervelmethoden is de vortonmethode, het onderwerp van het tweede deel van dit proefschrift.

Het wervelelement toegepast in de vortonmethode, is de vorton Dit kunnen we opvatten als een driedimensionale puntwervel. De deformatie van deze vortons volgt uit de zgn. Helmholtz vergelijking. De deformatie- en de verplaatsingsvergelijking voor de vortonen worden vervolgens numeriek opgelost. Er is echter over de afleiding van de deformatievergelijking discussie ontstaan in de literatuur. Uitgaande van twee in principe gelijkwaardige vormen van de Helmholtzvergelijking kwamen Novikov en Kuwabara tot twee niet-gelijkwaardige vortondeformatievergelijkingen. In dit proefschrift stellen wij een nieuwe afleiding van de vergelijking voor, die deze inconsistentie opheft en aantoont dat Novikovs noch Kuwabara's vergelijking aantrekkelijk is. Eén van de doelen van onze numerieke simulaties is geweest om de superioriteit van onze vergelijking aan te tonen; wij menen dat dit is gelukt; Kuwabara's vergelijking blijkt in elk geval onbruikbaar. Een ander doel van de simulaties is geweest het vergelijken van het gedrag van diverse wervelstructuren met dat van hun vorton-equivalenten. Wij hebben ons beperkt tot onderzoek naar het gedrag en de interactie van vortonringen, het vorton-equivalent van de wervelring. We waren uiteraard afhankelijk van de beschikbaarheid van experimentele, numerieke en analytische resultaten van anderen en van de mogelijkheden m.b.t. randvoorwaarden (alleen een *free-slip*-voorwaarde is mogelijk). Verder hebben we de volgende wervelfenomenen willen simuleren: werveldeformatie, wervelkerndeformatie, wervelreconnectie, en *alignment* van wervelbuizen. De configuraties die we hebben gesimuleerd zijn: een enkele vortonring (onderzoek naar kern en stabiliteit); een enkele pseudo-elliptische vortonring (deformatie en reconnectie); de botsing van twee coaxiale vortonringen (kerndeformatie en stabiliteit); de interactie van twee aanvankelijk parallel bewegende vortonringen (reconnectie); de interactie van twee "geknoopte" vortonringen (*alignment*); en een enkele vortonring in een afschuifstroming boven een vlakke plaat (een mogelijk eenvoudig model voor het gedrag van coherente structuren in een turbulente grenslaag).

Ook is de zgn. *vorton-division*-techniek onderzocht: het toevoegen van vortonen op plaatsen waar de afstand tussen naburige vortonen groter wordt dan een bepaalde waarde. De simulaties hebben laten zien dat in elk geval toevoegen zonder *updating* van de vortonen niet aanvaardbaar is. Met *updating* kan enige verbetering optreden, maar *division* met lineaire interpolatie blijkt geen remedie tegen ontsporende simulaties.

De simulaties hebben in elk geval laten zien dat aan de vortonring een kern(diameter) kan worden toegekend. De verdeling van de vorticiteit is echter niet homogeen verdeeld over de ring. Het aantal vortonen in de ring bepaalt de kerndiameter (en snelheid), zodat numerieke nauwkeurigheid niet met het aantal vortonen verbeterd kan worden (dit zou wellicht wel kunnen door een adnere modellering van de wervelring). Het feit dat de kerndiameter groeit bij wervelstrekking lijkt fysisch gezien onacceptabel. De discrete representatie van continue wervelstructuren is enerzijds de oorzaak van de mogelijkheid reconnectie te simuleren (bij dichte nadering van vortonen treedt een "bifurcatie" op naar een nieuwe stabiele situatie; de vraag is hoe "fysisch" dit is), aan de andere kant leidt het tot chaotisch gedrag van de vortonen bij dichte nadering. *Alignment* wordt goed gerepresenteerd, maar ook hier lopen de simulaties mis bij dichte nadering van de vortonen; in experimenten treedt waarschijnlijk annihilatie van vorticiteit op.

De toepassing van de vortonmethode bij het onderzoek naar coherente structuren in een grenslaag wordt gehinderd door de mogelijk belangrijke invloed van de *no-slip*-conditie aan de wand. Toch laat onze eenvoudige simulatie interessant gedrag van de vortonring zien en een grote piek in een grootheid die verband houd met de *Reynolds shear stress*. Dit duidt op het optreden van een zgn. *burst*; het optreden daarvan wordt in elk geval bepaald door de grootte van de snelheid buiten de grenslaag.

We kunnen concluderen dat de vortonmethode relatief eenvoudig is, weinig rekeninspanning vergt, en in bepaalde situaties goede simulatieresultaten oplevert. Wij moeten echter niet tveel van de methode verwachten, enerzijds vanwege de discrete representatie, anderzijds vanwege het ontbreken van visceus gedrag.

In de *Epilogue* worden tenslotte beide delen van het proefschrift weer bij elkaar gebracht opeen wetenschapsfilosofisch niveau. Zowel Kelvin bij de ontwikkeling van zijn wervelatoomtheorie als huidige onderzoekers die simulaties uitvoeren met *vortex methods* als de vortonmethode, stuiten op de beperkingen of onvolkomenheden van hun zgn. *computational model*. Samen met andere factoren was dit de oorzaak van de beperkte bloei van Kelvins atoomtheorie. Wat betreft onze simulatie van de vortonring in een *shear flow* boven een plaat: hierbij hebben we niet alleen te maken met een wellicht inadequaat *computational model*, maar we moeten ons ook afvragen of deze modellering van turbulente grenslaagstromingen zinvol en niet misleidend is.

# Levensloop

Ik werd geboren op 30 juli 1966 in Gouda en bracht daar mijn eerste levensjaren door. In de periode 1978-1984 bezocht ik het Fioretti College in Lisse en behaalde daar het diploma Gymnasium-$\beta$. Vervolgens studeerde ik in de periode 1984-1989 Werktuigbouwkunde aan de Technische Universiteit Delft. Na mijn afstuderen werd ik enige maanden door het Delfts Universiteitsfonds betaald tot een officiële aanstelling volgde als Onderzoeker in Opleiding in dienst van de Stichting FOM (FOM-werkgroep SWD-b). Mijn onderzoek vond plaats bij de vakgroep Stromingsleer van de Faculteit der Werktuigbouwkunde en Maritieme Techniek (Laboratorium voor Aero- en Hydrodynamica).